https://mchba.github.io/240731_intro_to_earsm.pdf

← Slides are here!

Introduction to EARS models

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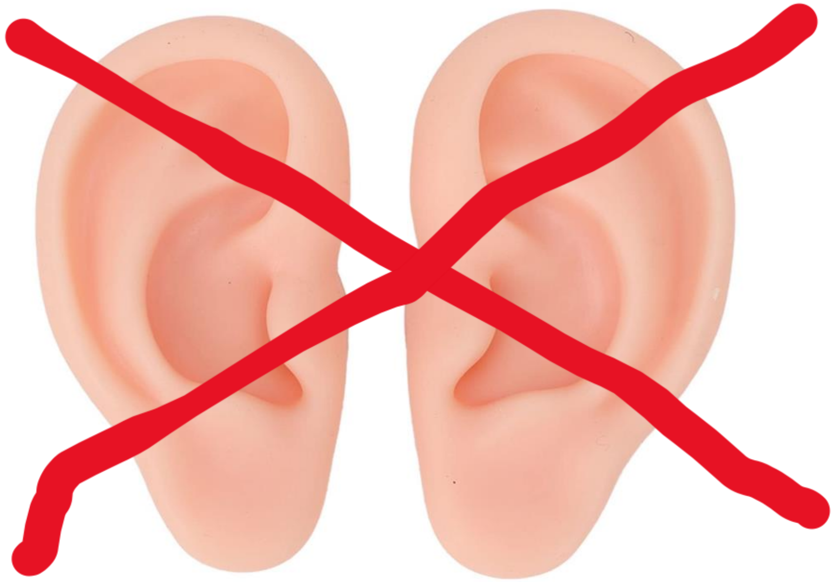
July 31, 2024

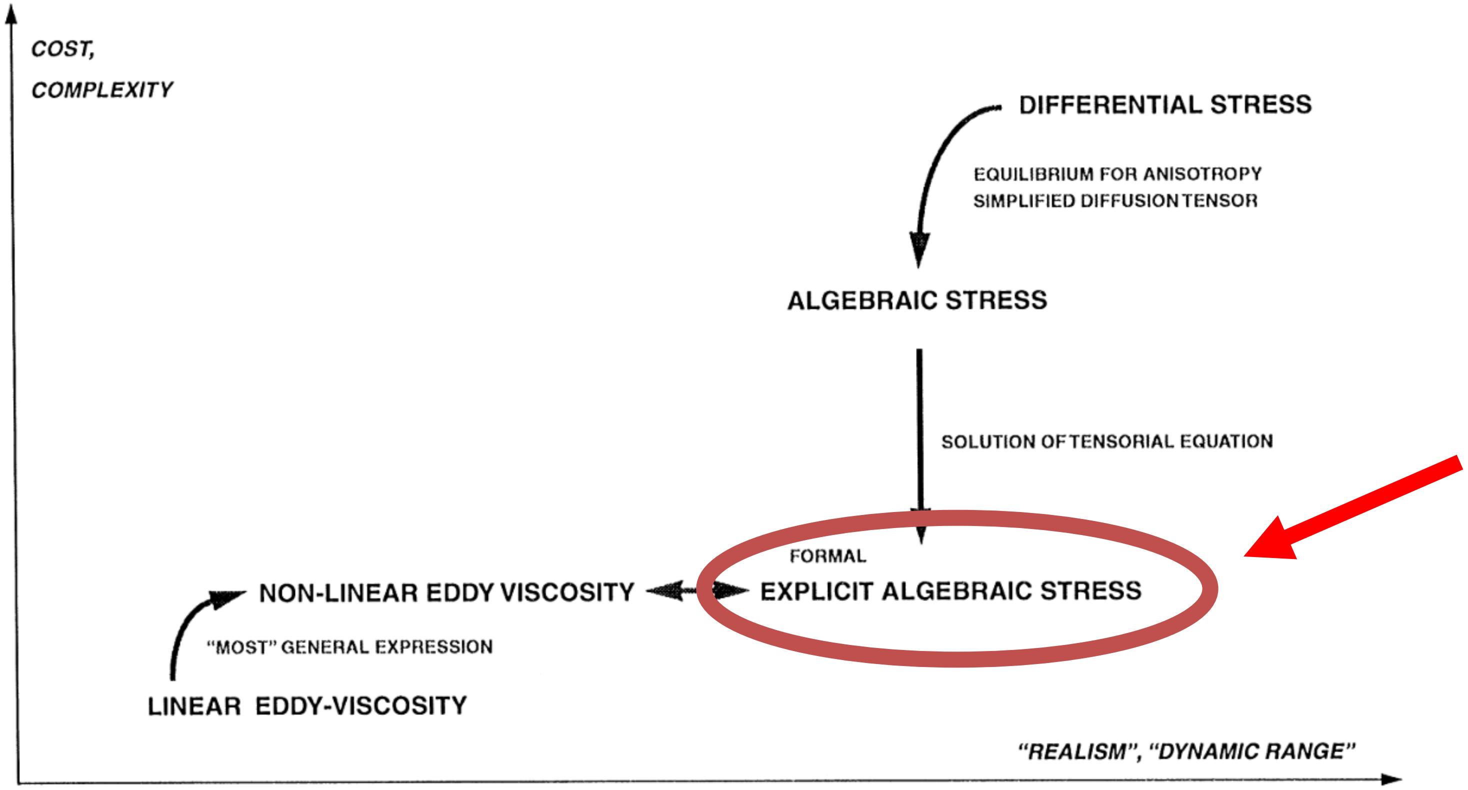
EARS



Explicit **A**lgebraic **R**eynolds **S**tress models ✓

What is an EARS model?

- A class of turbulence models for RANS



Gatski & Jongen (2000)

Example of an EARS model

- Model of Wallin & Johansson (2000) with *k*-*ε* platform.

$$\frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = \underbrace{-\overline{u_i' u_j'} \frac{\partial U_i}{\partial x_j}}_{\mathcal{P}} - \varepsilon + \underbrace{\frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j}\right)}_{\mathcal{D}^{(k)}},$$

$$\frac{\partial \varepsilon}{\partial t} + U_j \frac{\partial \varepsilon}{\partial x_j} = (C_{\varepsilon 1}\mathcal{P} - C_{\varepsilon 2}\varepsilon)\frac{\varepsilon}{k} + \underbrace{\frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_j}\right)}_{\mathcal{D}^{(\varepsilon)}}.$$

$$\mathbf{S} = S_{ij} = \frac{1}{2}\frac{k}{\varepsilon}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right),$$

$$\mathbf{\Omega} = \Omega_{ij} = \frac{1}{2}\frac{k}{\varepsilon}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right),$$

$$II_S = S_{ij}S_{ji} \qquad \mathbf{T}^{(1)} = \mathbf{S}$$

$$II_\Omega = \Omega_{ij}\Omega_{ji} \qquad \mathbf{T}^{(4)} = \mathbf{S\Omega} - \mathbf{\Omega S}$$

$$P_1 = \left(\frac{1}{27}c_1'^2 + \frac{9}{20}II_S - \frac{2}{3}II_\Omega\right)c_1',$$

$$P_2 = P_1^2 - \left(\frac{1}{9}c_1'^2 + \frac{9}{10}II_S + \frac{2}{3}II_\Omega\right)^3.$$

$$N = \begin{cases} \frac{c_1'}{3} + \left(P_1 + \sqrt{P_2}\right)^{1/3} + \operatorname{sign}\left(P_1 - \sqrt{P_2}\right)\left|P_1 - \sqrt{P_2}\right|^{1/3}, & P_2 \geq 0 \\ \frac{c_1'}{3} + 2\left(P_1^2 - P_2\right)^{1/6} + \cos\left(\frac{1}{3}\left(\frac{P_1}{\sqrt{P_1^2 - P_2}}\right)\right), & P_2 < 0 \end{cases}$$

$$\beta_1 = -\frac{6}{5}\frac{N}{N^2 - 2II_\Omega}, \qquad \beta_4 = -\frac{6}{5}\frac{1}{N^2 - 2II_\Omega}.$$

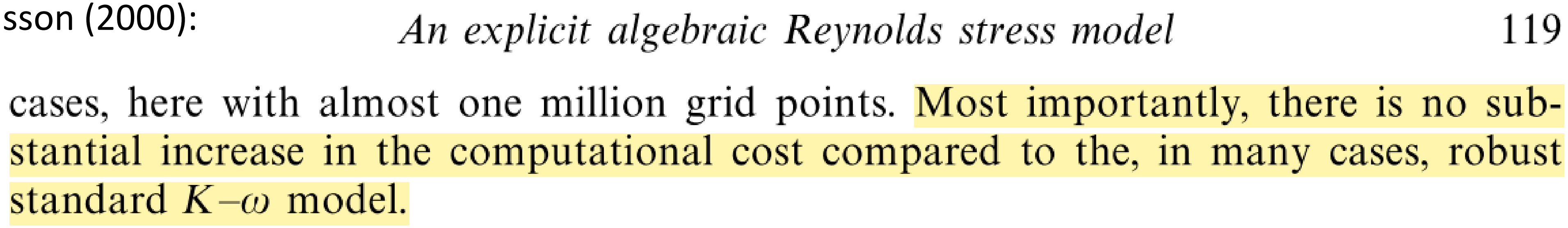
$$a_{ij} = \beta_1 T_{ij}^{(1)} + \beta_4 T_{ij}^{(4)}$$

$$\overline{u_i' u_j'} = k a_{ij} + \frac{2}{3}k\delta_{ij}$$

Why EARS models?

	Linear EVM	WJ-EARSM
Anisotropic freestream turbulence	❌	✅
Secondary flows	❌	✅
Counter-gradient heat fluxes	❌	✅
Realizable turbulence	Some	✅
Sensitive to rotation	Very few	✅

Wallin & Johansson (2000):



An explicit algebraic Reynolds stress model 119

cases, here with almost one million grid points. Most importantly, there is no substantial increase in the computational cost compared to the, in many cases, robust standard K–ω model.

How I got interested in EARS models

FLOW centre
KTH, Stockholm, Sweden

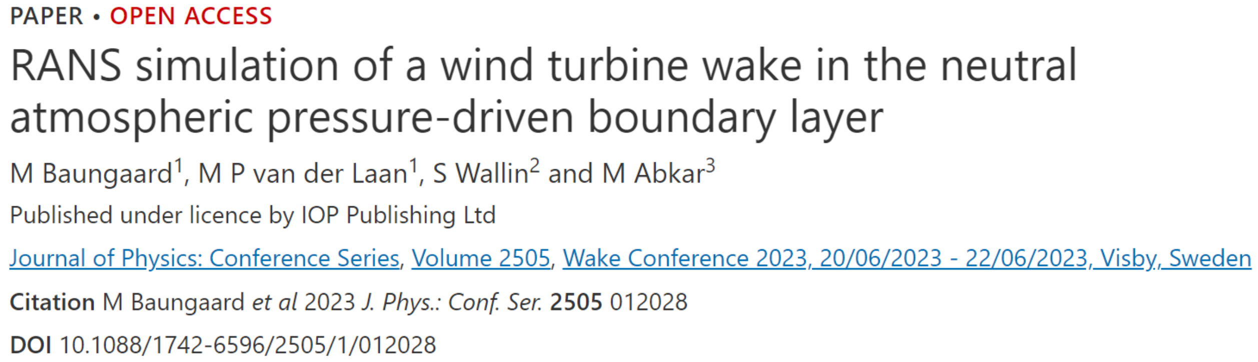
Visited dr. Stefan Wallin in Autumn 2021.

EARSM for wind applications

Research article | 🅭🅯 10 Oct 2022

Wind turbine wake simulation with explicit algebraic Reynolds stress modeling

Mads Baungaard ✉, Stefan Wallin, Maarten Paul van der Laan, and Mark Kelly



PAPER • OPEN ACCESS

RANS simulation of a wind turbine wake in the neutral atmospheric pressure-driven boundary layer

M Baungaard[1], M P van der Laan[1], S Wallin[2] and M Abkar[3]

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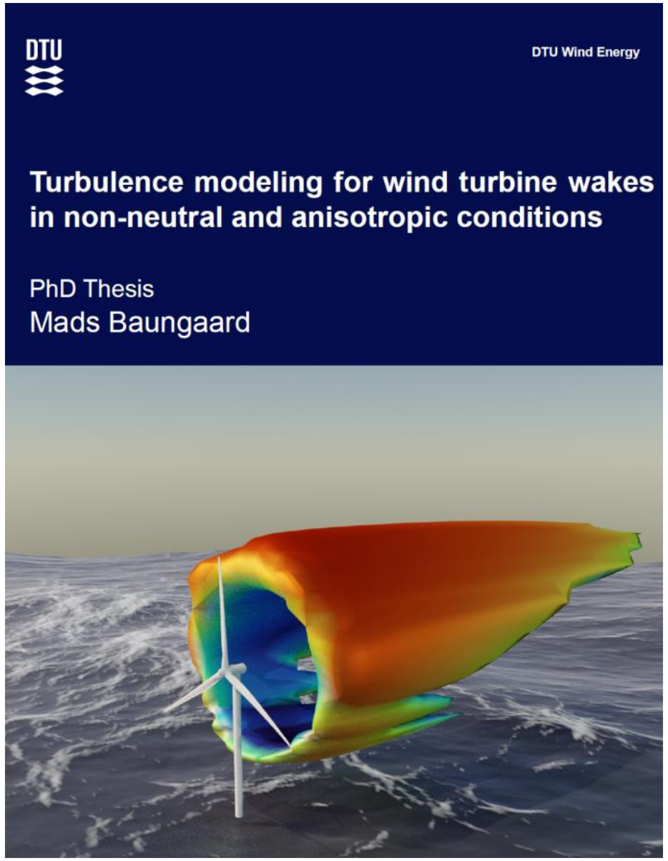


DTU

DTU Wind Energy

Turbulence modeling for wind turbine wakes in non-neutral and anisotropic conditions

PhD Thesis

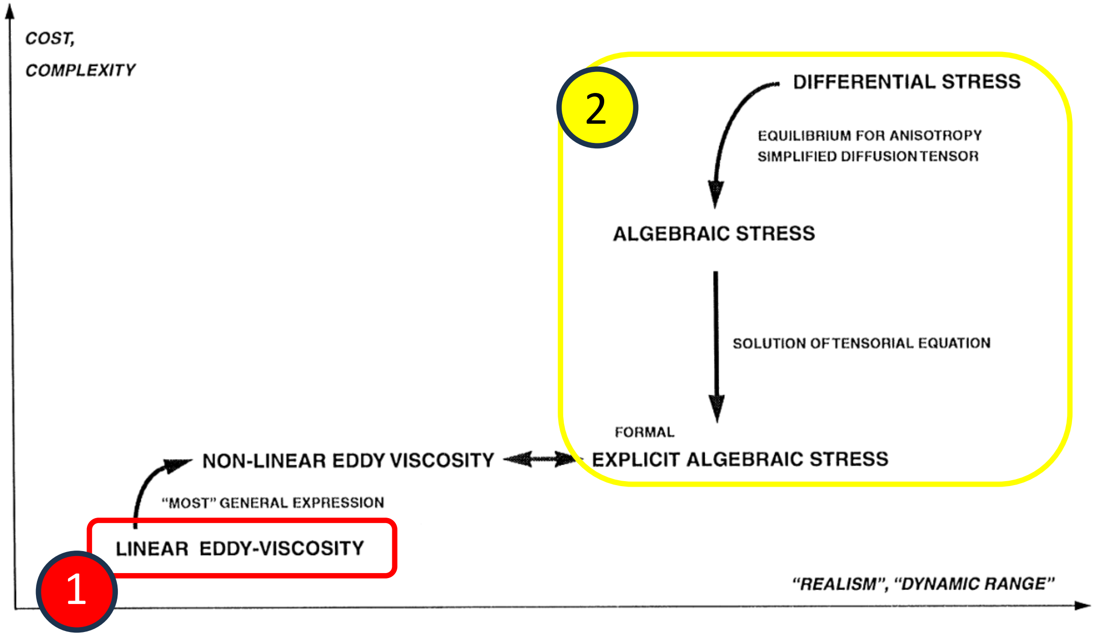
Mads Baungaard

Outline

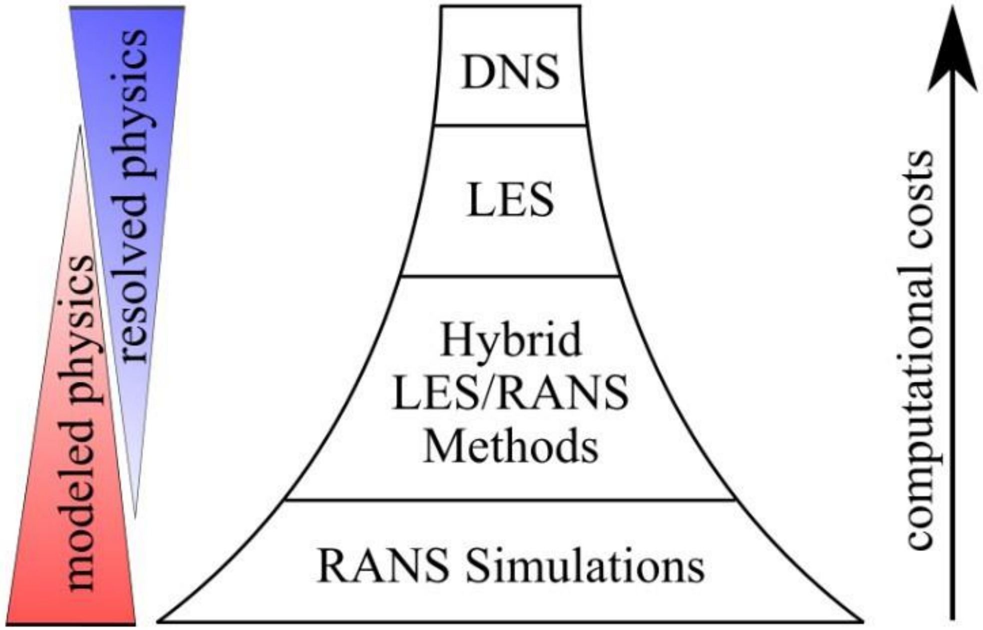
1 Linear eddy-viscosity models (EVMs)

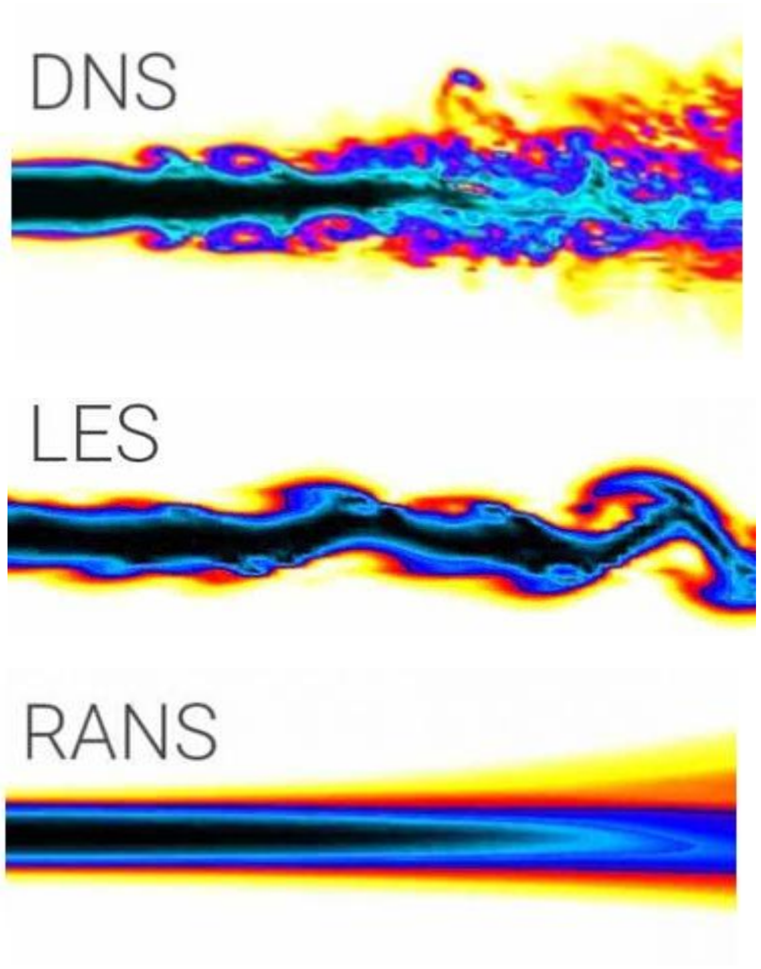
2 Theory of EARS models

3 Applications of EARS models



Computational fluid dynamics

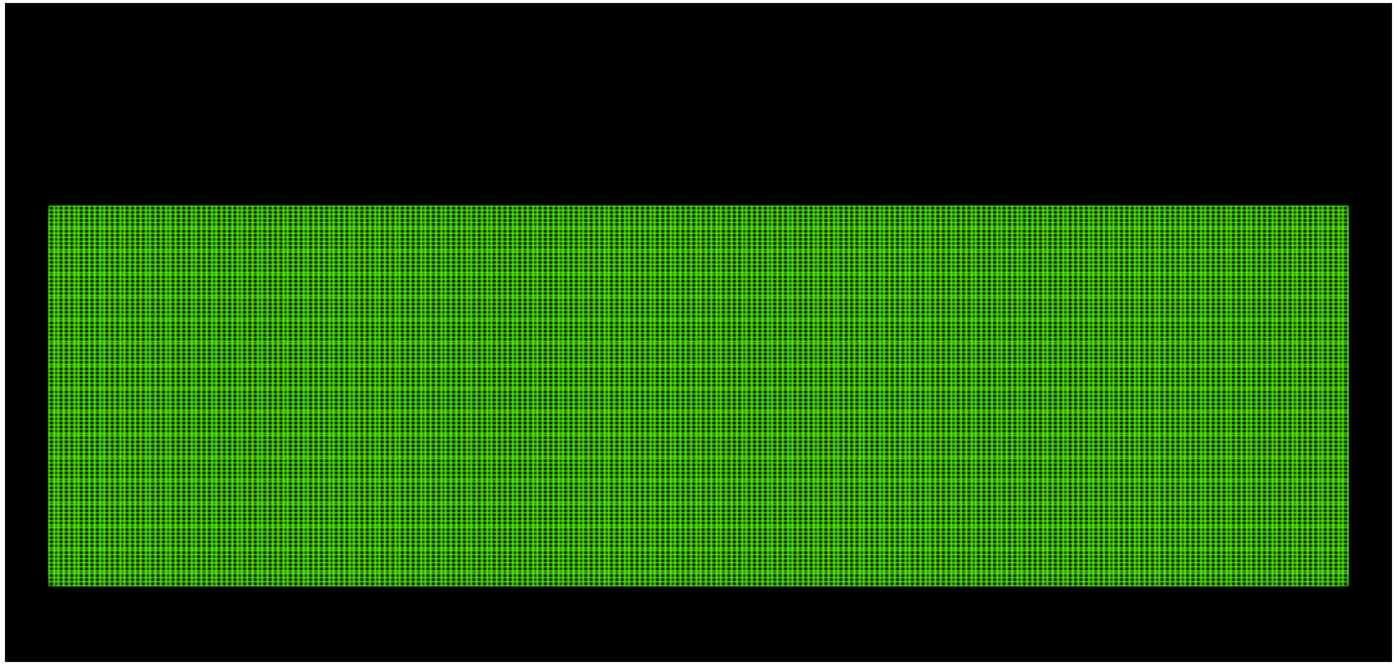


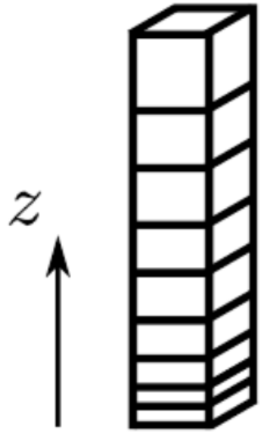


One method is not "better". They are just different tools.

Computational cost?

Example: simulation of conventionally neutral boundary layer (CNBL)





1D URANS (EllipSys1D):
25 core-seconds

LES (EllipSys3D): 4500 core-hours

$\mathcal{O}\left(10^6\right)$

Baungaard et al. (2024)

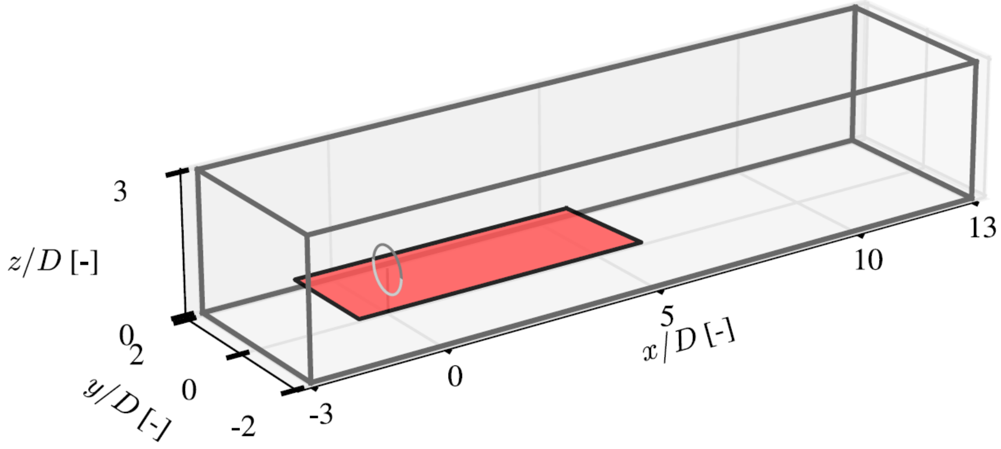
RANS equations

Turbulence modelling: *“How do we get the Reynolds stress tensor?”*

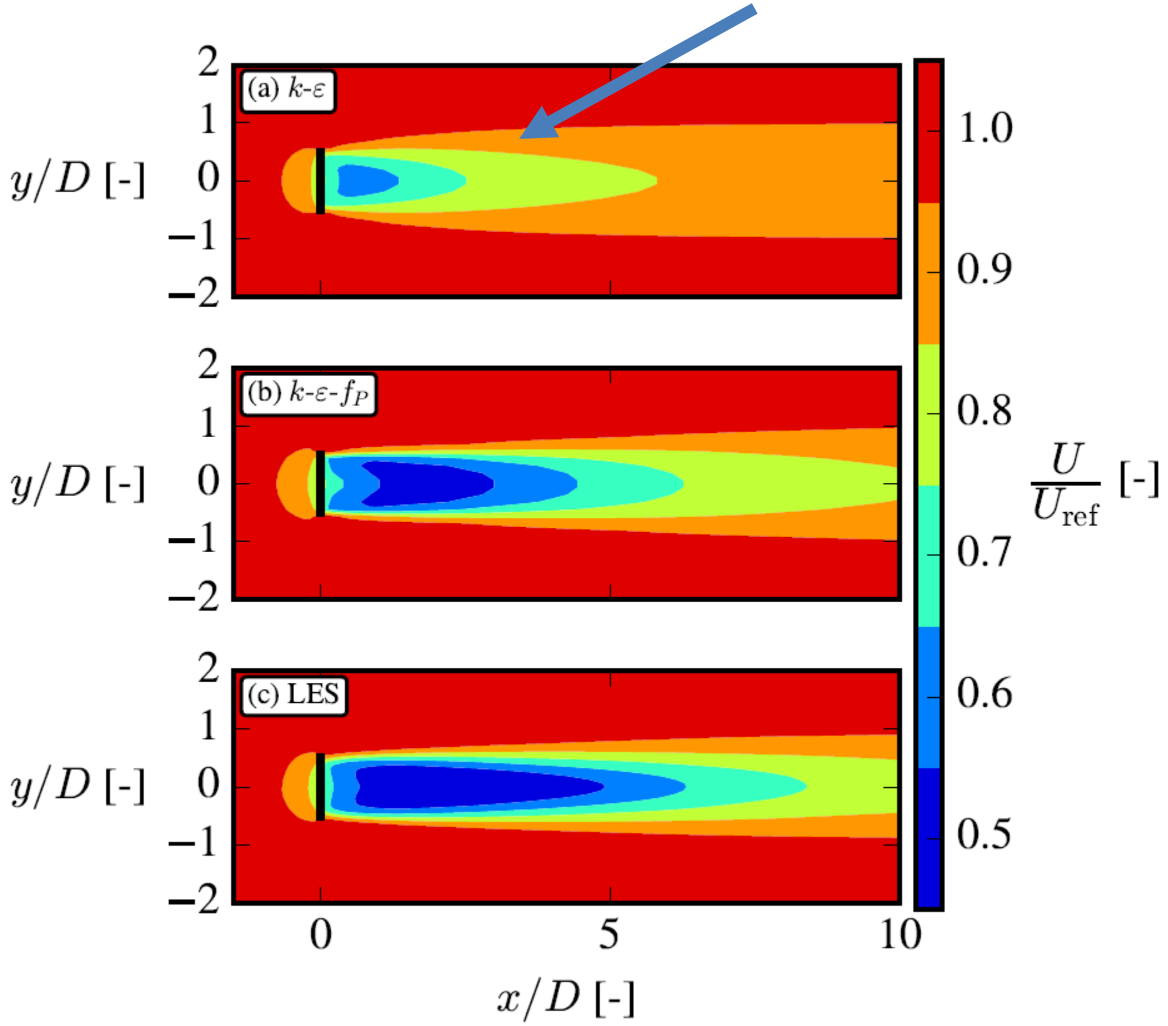
$$\frac{\partial U_i}{\partial t} + \frac{\partial \left(U_j U_i\right)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \boxed{\overline{u'_i u'_j}}\right) + F_i$$

Very important for getting good results!

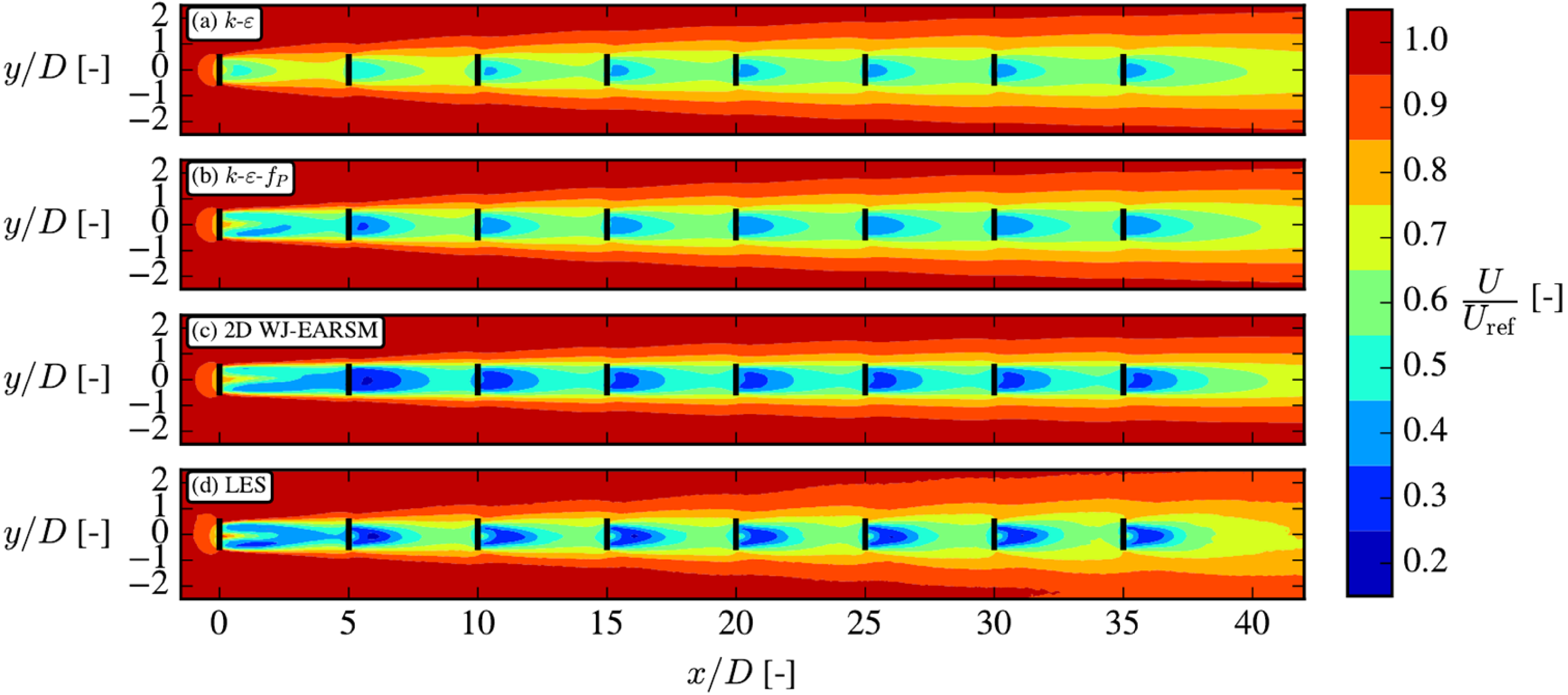
Example 1: a wind turbine wake



Standard k-eps is over-diffusive.



Example 2: a row of turbines



Message

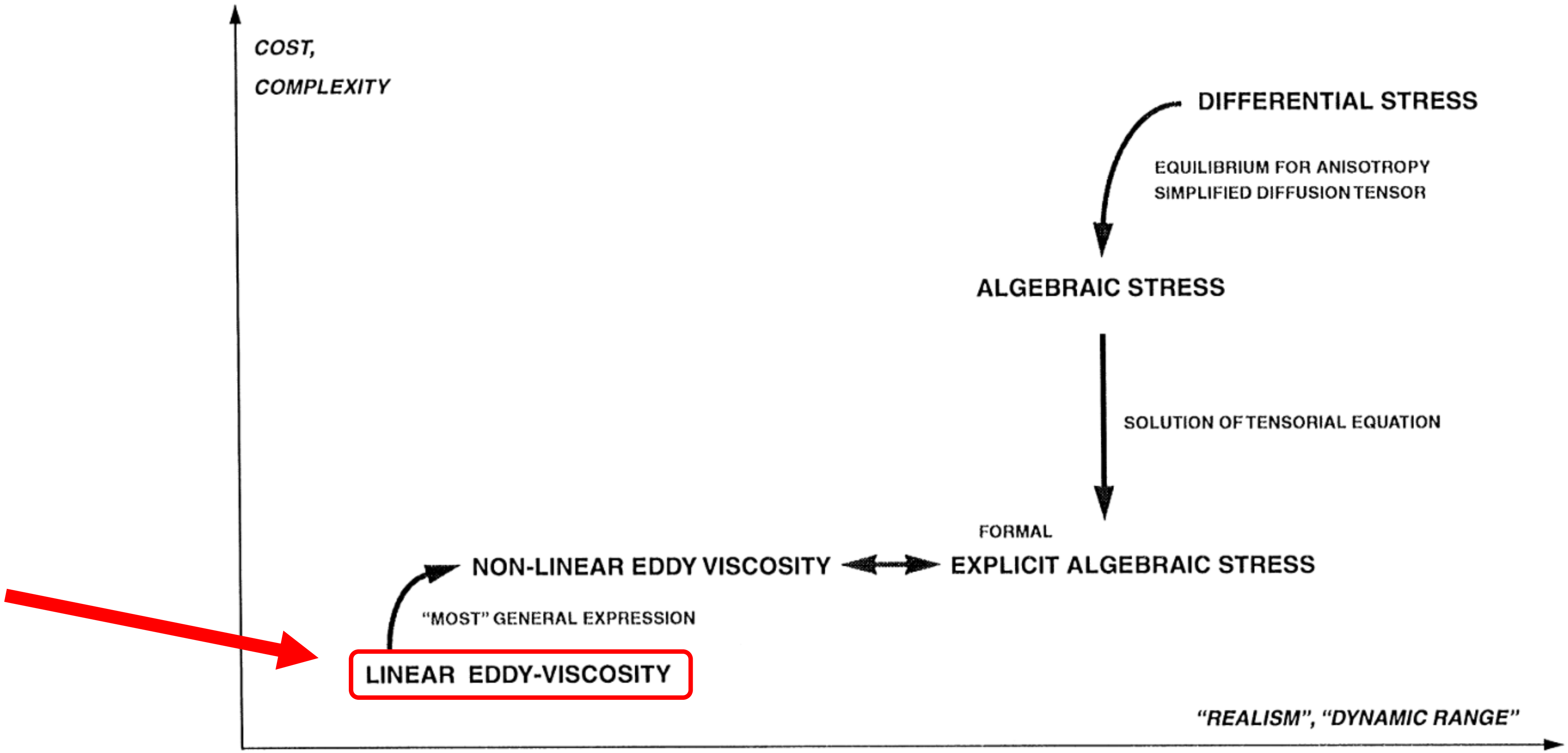
Before moving to expensive LES and DNS:

> *"What is the best we can do with RANS?"*

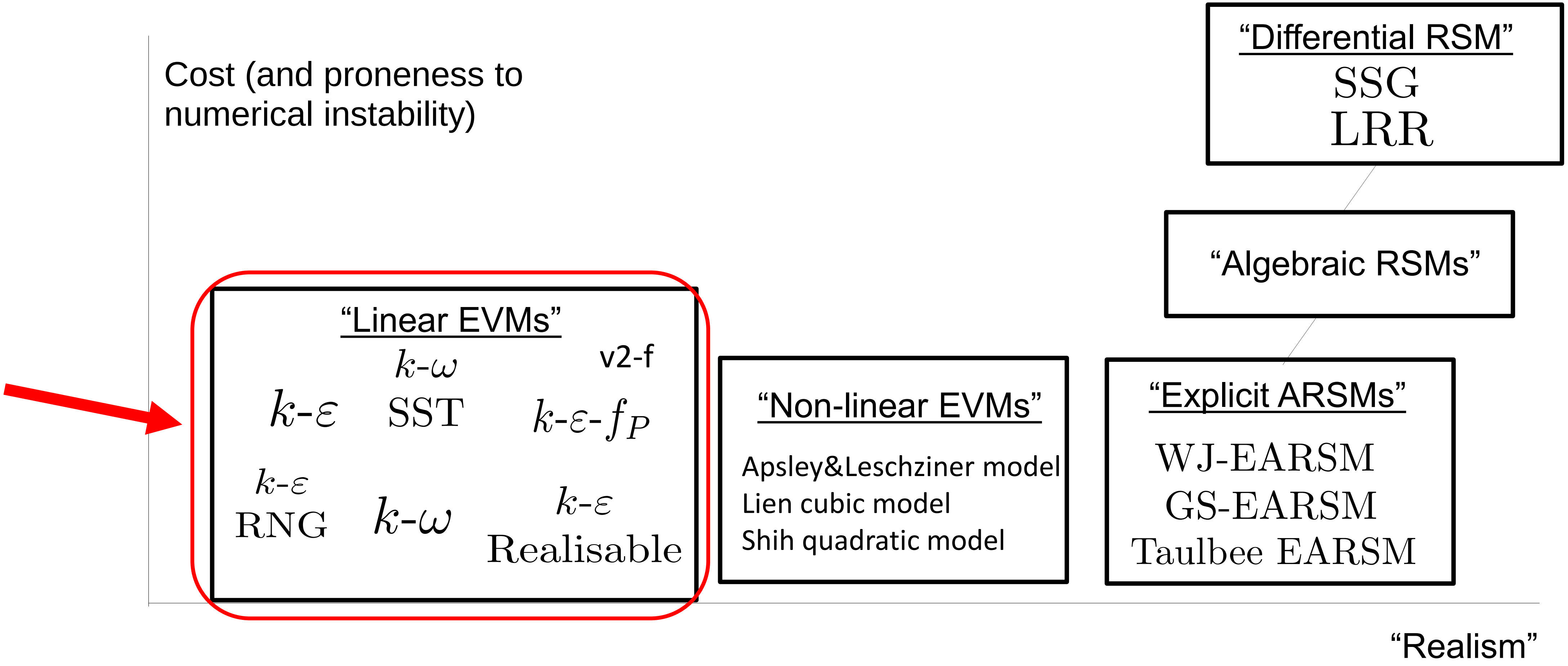
Shift weight to the turbulence modeller

Linear eddy-viscosity models (EVMs)

How to model uiuj?



How to model uiuj?



Cost (and proneness to numerical instability)

"Realism"

"Linear EVMs"

k-ε

k-ω SST

v2-f

k-ε-f_P

k-ε RNG

k-ω

k-ε Realisable

"Non-linear EVMs"

Apsley&Leschziner model
Lien cubic model
Shih quadratic model

"Explicit ARSMs"

WJ-EARSM
GS-EARSM
Taulbee EARSM

"Algebraic RSMs"

"Differential RSM"

SSG
LRR

Linear eddy-viscosity models (EVMs)

$$\boxed{\frac{\partial U_i}{\partial t} + \frac{\partial \left(U_j U_i\right)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \overline{u_i' u_j'}\right) + F_i}$$

Idea: Maybe the red term is similar to the blue term.

$$\Rightarrow \overline{u_i' u_j'} = -\nu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

Symmetric ✓

$\overline{u_i' u_i'} = 0$ ✗

Idea 2: The trace should be equal to 2 times TKE

$$\boxed{\overline{u_i' u_j'} = -\nu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) + \frac{2}{3} k \delta_{ij}}$$

Symmetric ✓

$\overline{u_i' u_i'} = 2k$ ✓

The Boussinesq hypothesis

All linear EVMs use the Boussinesq hypothesis as their *constitutive relation*.

$$\boxed{\overline{u_i' u_j'} = -\nu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) + \frac{2}{3} k \delta_{ij}}$$

They only differ in how they obtain the eddy viscosity.

$$\nu_t \equiv C_\mu^{(\text{eff})} k \tau$$

Model	$C_\mu^{(\text{eff})}$	τ
k-ε	C_μ	$\frac{k}{\varepsilon}$
k-ε-f_P	$C_\mu f_P$	$\frac{k}{\varepsilon}$
k-ω	β^*	$\frac{1}{\beta^* \omega}$
k-ω SST	$\min\left(\beta^*, \frac{a_1}{\sqrt{-2 II_\Omega} F_2}\right)$	$\frac{1}{\beta^* \omega}$

What is meant by *linear* EVM?

$$\boxed{\overline{u_i'u_j'} = -\nu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) + \frac{2}{3}k\delta_{ij}}$$

$$a_{ij} \equiv \frac{\overline{u_i'u_j'}}{k} - \frac{2}{3}\delta_{ij},$$

$$S_{ij} \equiv \frac{1}{2}\tau\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

$$\nu_t \equiv C_\mu^{(\mathrm{eff})} k\tau$$

$$\boxed{a_{ij} = -2C_\mu^{(\mathrm{eff})} S_{ij}}$$

Anisotropy tensor is linear in the strain rate tensor.

Deficiency of linear EVMs

$$\boxed{a_{ij} = -2C_{\mu}^{(\text{eff})} S_{ij}}$$

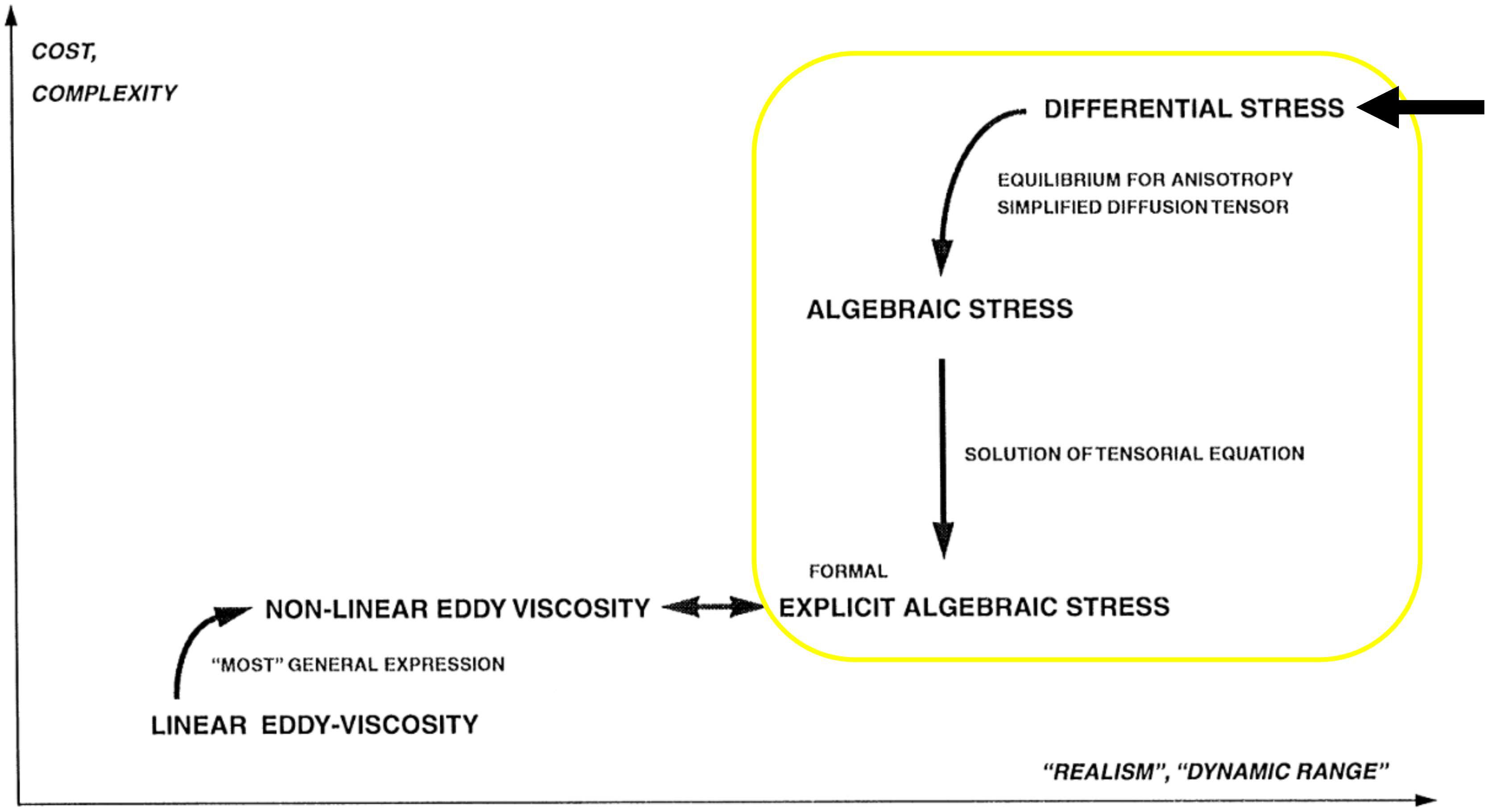
- "Blind" to the antisymmetric part of the velocity gradient tensor $\tau \frac{\partial U_i}{\partial x_j} = S_{ij} + \Omega_{ij}$ which is important for flows with curvature, swirl or rotation.
- For horizontally homogeneous flows, linear EVMs give $a_{11} = a_{22} = a_{33} = 0$
- Directions of a_{ij} and S_{ij} are always aligned
- Realisibility not ensured
- …

Lots of ad-hoc modifications have been suggested to fix these problems!

Theory of EARS models

Warning: lots of equations!

How to model $u_i u_j$?



Start here

The starting point for EARS models

- An exact equation

$$\frac{D\overline{u_i'u_j'}}{Dt} = \underbrace{-\overline{u_j'u_k'}\frac{\partial U_i}{\partial x_k} - \overline{u_i'u_k'}\frac{\partial U_j}{\partial x_k}}_{\mathcal{P}_{ij}} \underbrace{-\frac{\partial \overline{u_j'u_i'u_k'}}{\partial x_k} + \nu\frac{\partial^2\overline{u_i'u_j'}}{\partial x_k^2}}_{\mathcal{D}_{ij}^t+\mathcal{D}_{ij}^v} \underbrace{-\frac{1}{\rho}\overline{u_j'\frac{\partial p'}{\partial x_i}} - \frac{1}{\rho}\overline{u_i'\frac{\partial p'}{\partial x_j}}}_{vel\ pgrad\ corr} - \underbrace{2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}}_{\varepsilon_{ij}}$$

Terms in red need to be modelled

A more compact form of the uiuj equation

- A typical decomposition

$$\underbrace{-\frac{1}{\rho}\overline{u_j'\frac{\partial p'}{\partial x_i}}-\frac{1}{\rho}\overline{u_i'\frac{\partial p'}{\partial x_j}}}_{vel\ pgrad\ corr}=\underbrace{\frac{1}{\rho}\overline{p'\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)}}_{\Pi_{ij}}\underbrace{-\frac{1}{\rho}\left(\frac{\partial\overline{p'u_j'}}{\partial x_i}+\frac{\partial\overline{p'u_i'}}{\partial x_j}\right)}_{\mathcal{D}_{ij}^p}$$

and a typical model assumption

$$\varepsilon_{ij}=\frac{2}{3}\varepsilon\delta_{ij},$$

allow the uiuj equation to be written as:

$$\boxed{\frac{D\overline{u_i'u_j'}}{Dt}=\mathcal{P}_{ij}+\mathcal{D}_{ij}+\Pi_{ij}-\frac{2}{3}\varepsilon\delta_{ij}}.$$

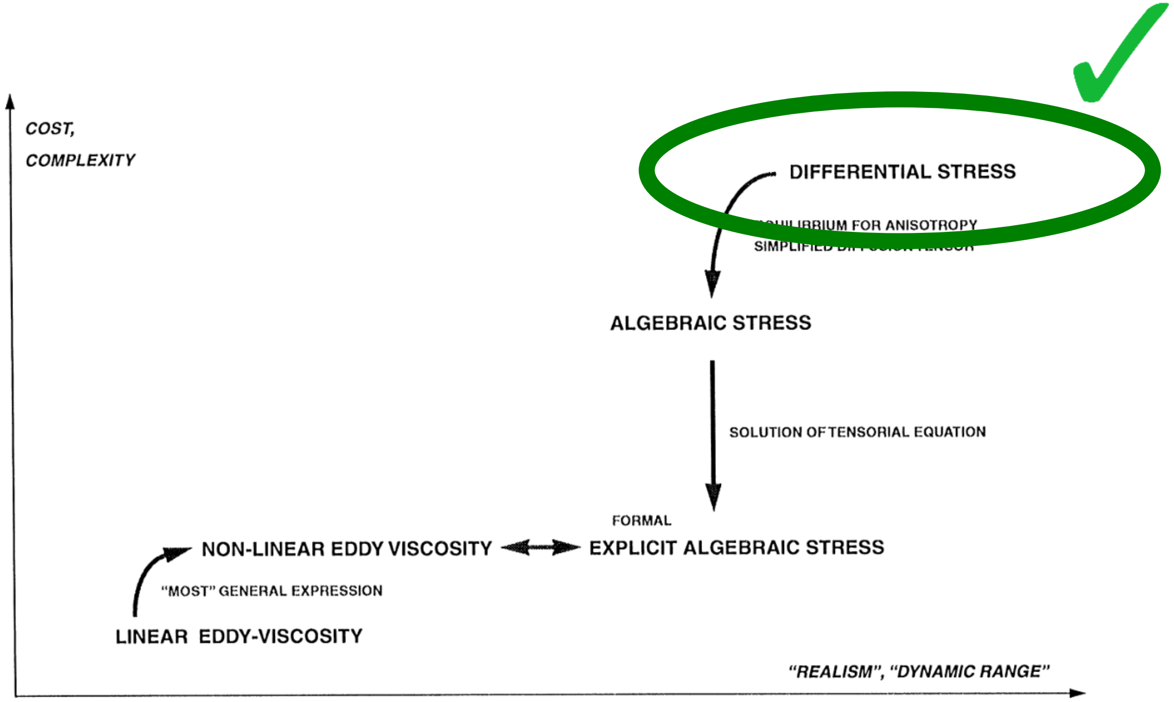
Differential Reynolds stress models (DRSMs)

$$\boxed{\frac{D\overline{u_i'u_j'}}{Dt} = \mathcal{P}_{ij} + \mathcal{D}_{ij} + \Pi_{ij} - \frac{2}{3}\varepsilon\delta_{ij}}.$$

- 7 equations (6 for u_iu_j + 1 for ε)
- Need models for

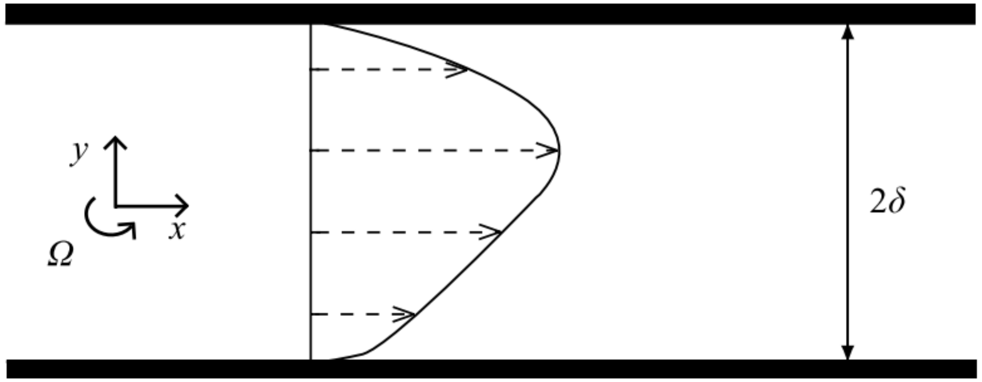
$$\Pi_{ij} \qquad \mathcal{D}^p_{ij} \qquad \mathcal{D}^t_{ij}$$

- Complicated, but possible. Two models (LRR and SSG) are available in OpenFOAM.

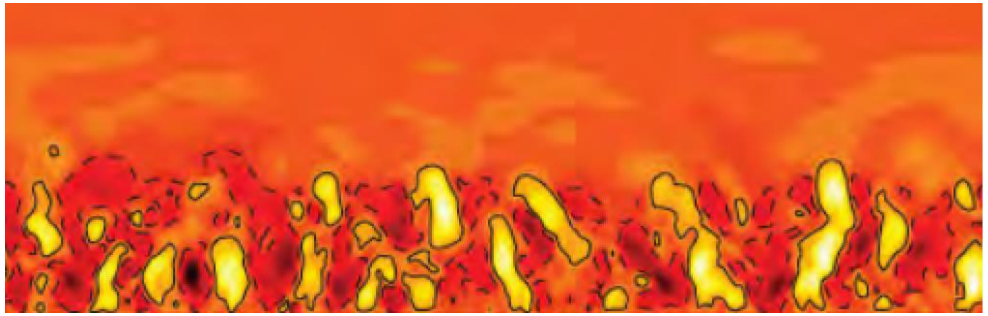


Example: rotating channel flow

- Velocity profile is asymmetric

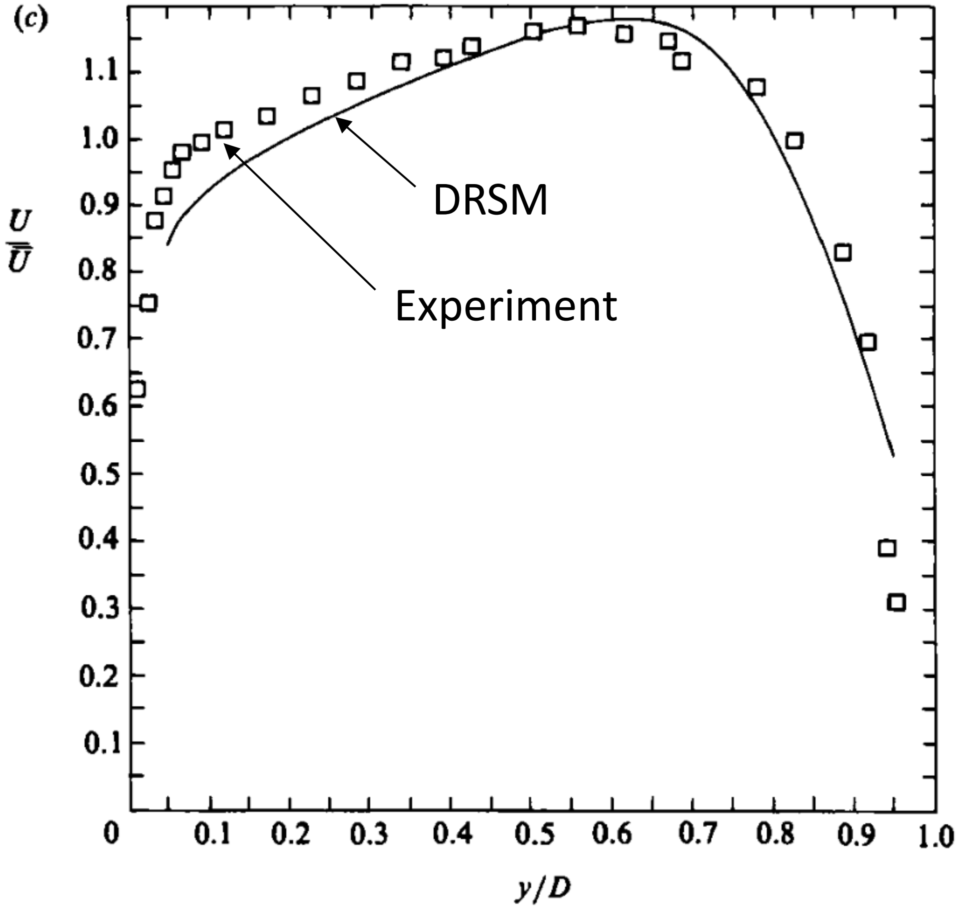


- Turbulence activity also asymmetric



DNS from Grundenstam et al. (2008)

Launder et al. (1987)



From *differential* RSM to *algebraic* RSM (ARSM)

Advantages of DRSMs:

✓ More physics

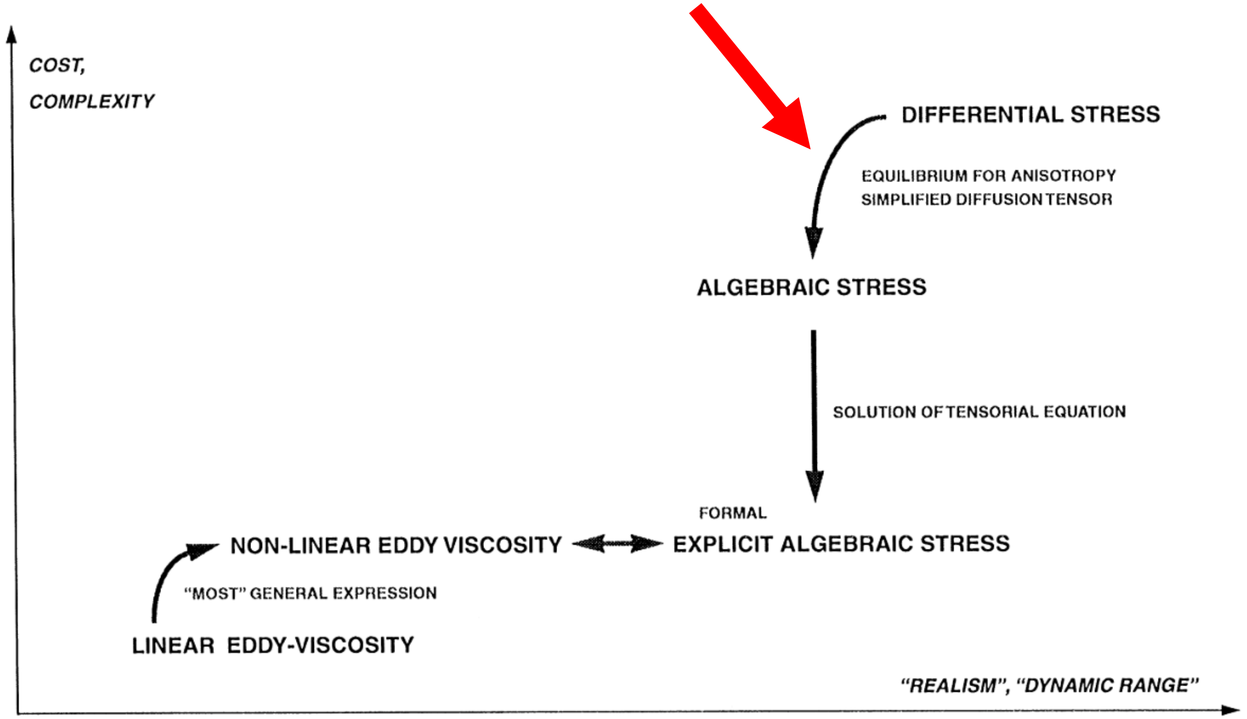
Disadvantages of DRSMs:

- ❌ Expensive (7 eqs)
- ❌ Difficult to implement
- ❌ Numerical robustness not good

Idea by Rodi (1972,1976):

"Transform the DRSM equations into a set of algebraic equations."

Next step



$$\mathbf{a} = a_{ij} \equiv \frac{\overline{u_i'u_j'}}{k} - \frac{2}{3}\delta_{ij},$$

Derivation of the ARSM equations 1/3

- Rewrite DRSM equations to

$$\frac{Da_{ij}}{Dt} = \frac{1}{k}\left(\mathcal{P}_{ij} + \Pi_{ij} + \mathcal{D}_{ij} - \frac{2}{3}\varepsilon\delta_{ij} - \left(\frac{\overline{u_i'u_j'}}{k}\right)\left(\mathcal{P} + \mathcal{D}^{(k)} - \varepsilon\right)\right)$$

and rearrange

“Weak-equilibrium assumption”

$$\Rightarrow \frac{Da_{ij}}{Dt} - \cancelto{0}{\underbrace{\left(\frac{\mathcal{D}_{ij}}{k} - \frac{\overline{u_i'u_j'}}{k^2}\mathcal{D}^{(k)}\right)}_{\mathcal{D}^{(a)}_{ij}}} = \frac{1}{k}\left(\mathcal{P}_{ij} + \Pi_{ij} - \frac{2}{3}\varepsilon\delta_{ij} - \left(\frac{\overline{u_i'u_j'}}{k}\right)(\mathcal{P} - \varepsilon)\right)$$

Derivation of the ARSM equations 2/3

- The "weak-equilibrium assumption" gives an *algebraic* set of equations

$$0 = \mathcal{P}_{ij} + \Pi_{ij} - \frac{2}{3}\varepsilon\delta_{ij} - \left(\frac{\overline{u_i' u_j'}}{k}\right)(\mathcal{P} - \varepsilon)$$

and rewrite to

$$\boxed{a_{ij}\left(\frac{\mathcal{P}}{\varepsilon} - 1\right) = \frac{\mathcal{P}_{ij}}{\varepsilon} + \frac{\Pi_{ij}}{\varepsilon} - \frac{2}{3}\frac{\mathcal{P}}{\varepsilon}\delta_{ij}}$$

Derivation of the ARSM equations 3/3

$$\boxed{a_{ij}\left(\frac{\mathcal{P}}{\varepsilon}-1\right)=\frac{\mathcal{P}_{ij}}{\varepsilon}+\frac{\Pi_{ij}}{\varepsilon}-\frac{2}{3}\frac{\mathcal{P}}{\varepsilon}\delta_{ij}}$$

- Shear production definitions (exact)

$$\frac{\mathcal{P}_{ij}}{\varepsilon}=-a_{kj}\left(S_{ik}+\Omega_{ik}\right)-a_{ik}\left(S_{kj}-\Omega_{kj}\right)-\frac{4}{3}S_{ij} \qquad \frac{\mathcal{P}}{\varepsilon}=-a_{ik}S_{ki}$$

- Launder-Reece-Rodi (LRR) pressure-strain model

$$\frac{\Pi_{ij}}{\varepsilon}=-c_1a_{ij}+\frac{4}{5}S_{ij}+\frac{9c_2+6}{11}\left(a_{ik}S_{kj}+S_{ik}a_{kj}-\frac{2}{3}a_{km}S_{mk}\delta_{ij}\right)+\frac{7c_2-10}{11}\left(a_{ik}\Omega_{kj}-\Omega_{ik}a_{kj}\right)$$

LRR-ARSM

$$\boxed{\left(\frac{\mathcal{P}}{\varepsilon}-1+c_1\right)\mathbf{a}=-\frac{8}{15}\mathbf{S}+\frac{9c_2-5}{11}\left(\mathbf{aS}+\mathbf{Sa}-\frac{2}{3}\mathrm{tr}\left\{\mathbf{aS}\right\}\mathbf{I}\right)+\frac{7c_2+1}{11}\left(\mathbf{a\Omega}-\mathbf{\Omega a}\right)}$$

Simplification used by Taulbee (1992).
DNS by Shih & Shabbir (1993) support it.

→ If c_2=5/9

Simplified LRR-ARSM

$$\boxed{\underbrace{\frac{9}{4}\left(\frac{\mathcal{P}}{\varepsilon}-1+c_1\right)}_{N}\mathbf{a}=-\frac{6}{5}\mathbf{S}+\left(\mathbf{a\Omega}-\mathbf{\Omega a}\right)}$$

Simplified LRR-ARSM

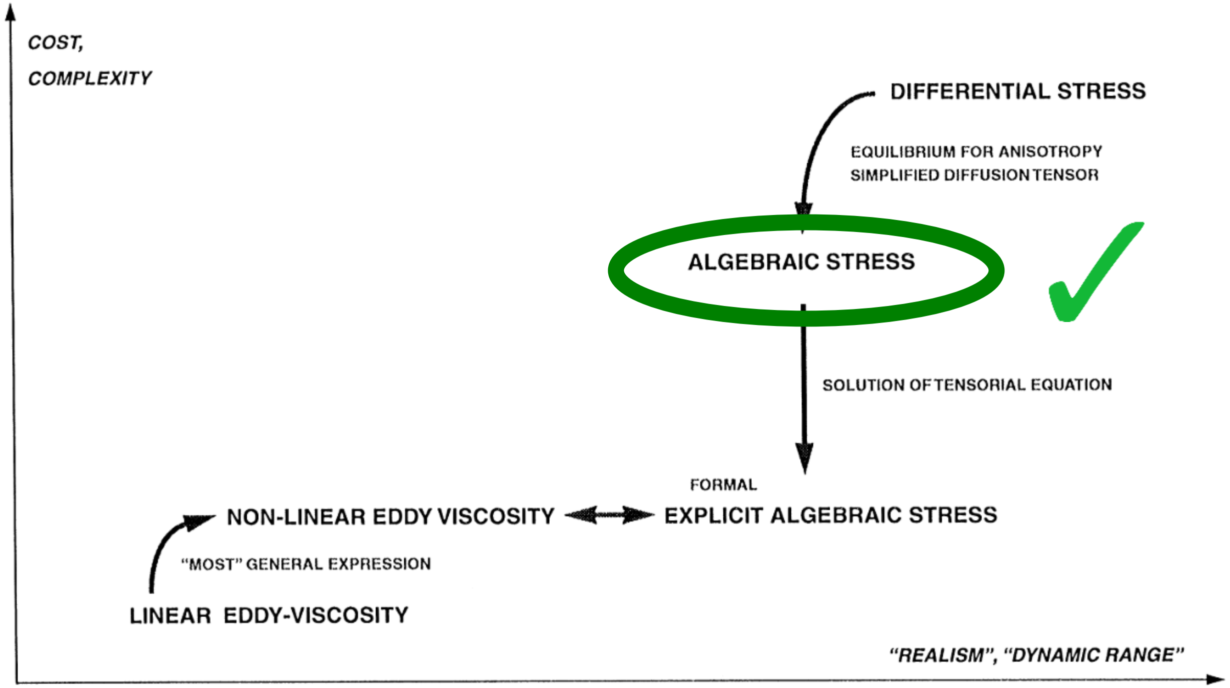
$$\boxed{N\mathbf{a} = -\frac{6}{5}\mathbf{S} + (\mathbf{a}\boldsymbol{\Omega} - \boldsymbol{\Omega}\mathbf{a})}$$

Only assumptions:

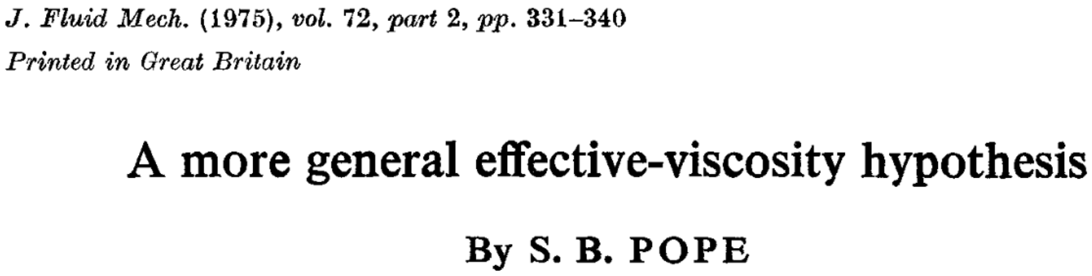
- Isotropic dissipation
- Weak-equilibrium assumption
- The LRR pressure-strain model with c_2=5/9

Problems:

1. Implicit equation for **a**
2. Non-linear in **a**



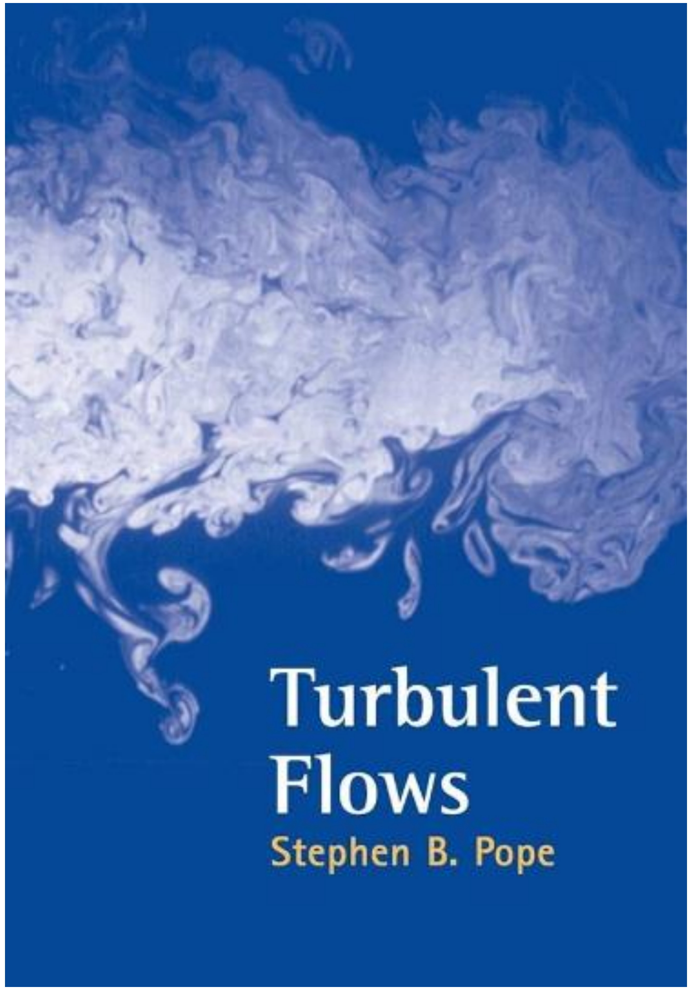
From *algebraic* RSM to *explicit algebraic* RSM



J. Fluid Mech. (1975), *vol.* 72, *part* 2, *pp.* 331–340
Printed in Great Britain

A more general effective-viscosity hypothesis

By S. B. POPE



Idea of Pope (1975):

1. Expand anisotropy tensor
2. Insert expansion in ARSM and simplify
3. Solve for expansion coefficients

In the mid-90s:

(Optional 4.) Ensure self-consistency

Step 1: anisotropy expansion

$$a_{ij} = \sum_{l=1}^{\infty} \beta_l T_{ij}^{(l)}$$

The anisotropy tensor must be:

- Dimensionless
- Galilean invariant
- Symmetric
- Traceless

..and so must the basis tensors too.

Basis candidates	Sym.	Tr. less
$\mathbf{S}$	✓	✓
$\mathbf{\Omega}$	✗	✓
$\mathbf{S}^2$	✓	✗
$\mathbf{S}^2 - \frac{1}{3}\mathrm{tr}\{\mathbf{S}^2\}\mathbf{I}$	✓	✓
$\mathbf{S\Omega}$	✗	✓
$\mathbf{S\Omega} - \mathbf{\Omega S}$	✓	✓

A complete tensor basis for statistically two-dimensional flows:

$$\mathbf{a} = \beta_1 \underbrace{\mathbf{S}}_{\mathbf{T}^{(1)}} + \beta_2 \underbrace{\left(\mathbf{S}^2 - \frac{1}{3}\mathrm{tr}\{\mathbf{S}^2\}\right)}_{\mathbf{T}^{(2)}} + \beta_4 \underbrace{(\mathbf{S\Omega} - \mathbf{\Omega S})}_{\mathbf{T}^{(4)}}$$

Step 2: insert and simplify

$$\boxed{N\mathbf{a} = -\frac{6}{5}\mathbf{S} + (\mathbf{a}\boldsymbol{\Omega} - \boldsymbol{\Omega}\mathbf{a})}$$

Insert expansion of **a** (trivial)

$$N\left(\beta_1\mathbf{T}^{(1)} + \beta_2\mathbf{T}^{(2)} + \beta_4\mathbf{T}^{(4)}\right) = -\frac{6}{5}\mathbf{S} + \left(\beta_1\mathbf{T}^{(1)} + \beta_2\mathbf{T}^{(2)} + \beta_4\mathbf{T}^{(4)}\right)\boldsymbol{\Omega} - \boldsymbol{\Omega}\left(\beta_1\mathbf{T}^{(1)} + \beta_2\mathbf{T}^{(2)} + \beta_4\mathbf{T}^{(4)}\right)$$

Simplify RHS (non-trivial)

$$N\left(\beta_1\mathbf{T}^{(1)} + \beta_2\mathbf{T}^{(2)} + \beta_4\mathbf{T}^{(4)}\right) = \left(-\frac{6}{5} + 2\beta_4 II_\Omega\right)\mathbf{T}^{(1)} + \beta_1\mathbf{T}^{(4)}$$

Step 3: finding the coefficients

$$N\left(\beta_1\mathbf{T}^{(1)}+\beta_2\mathbf{T}^{(2)}+\beta_4\mathbf{T}^{(4)}\right)=\left(-\frac{6}{5}+2\beta_4 II_\Omega\right)\mathbf{T}^{(1)}+\beta_1\mathbf{T}^{(4)}$$

Equate coefficients

$$N\beta_1=\left(-\frac{6}{5}+2\beta_4 II_\Omega\right),$$

$$N\beta_2=0,$$

$$N\beta_4=\beta_1,$$

Assume *N* is known and solve the 3x3 system of equations

$$\beta_1=-\frac{6}{5}\frac{N}{N^2-2II_\Omega},$$

$$\beta_2=0,$$

$$\beta_4=-\frac{6}{5}\frac{1}{N^2-2II_\Omega}.$$

$$\boxed{\mathbf{a}=\underbrace{-\frac{6}{5}\frac{N}{N^2-2II_\Omega}}_{\beta_1}\mathbf{T}^{(1)}\underbrace{-\frac{6}{5}\frac{1}{N^2-2II_\Omega}}_{\beta_4}\mathbf{T}^{(4)}}$$

Step 4: self-consistency (1/2)

$$\boxed{\mathbf{a} = \underbrace{-\frac{6}{5}\frac{N}{N^2 - 2II_\Omega}}_{\beta_1}\mathbf{T}^{(1)} \underbrace{-\frac{6}{5}\frac{1}{N^2 - 2II_\Omega}}_{\beta_4}\mathbf{T}^{(4)}}$$

- "Self-consistency" = No approximations from ARSM to EARSM.

→ We need a *N,* such that the above equation satisfies the simplified ARSM.

$$\begin{aligned} N &\equiv \frac{9}{4}\left(\frac{\mathcal{P}}{\varepsilon} - 1 + c_1\right) \\ &= \frac{9}{4}\left(-\mathrm{tr}\{\mathbf{aS}\} - 1 + c_1\right) \\ &= \frac{9}{4}(c_1 - 1) + \frac{27}{10}\frac{N II_S}{N^2 - 2II_\Omega} \end{aligned}$$

→ A cubic polynomial for *N*

Step 4: self-consistency (2/2)

Johansson & Wallin (1996): the real and positive root is

$$N = \begin{cases} \frac{c_1'}{3} + \left(P_1 + \sqrt{P_2}\right)^{1/3} + \mathrm{sign}\left(P_1 - \sqrt{P_2}\right)\left|P_1 - \sqrt{P_2}\right|^{1/3}, & P_2 \geq 0 \\ \frac{c_1'}{3} + 2\left(P_1^2 - P_2\right)^{1/6} + \cos\left(\frac{1}{3}\left(\frac{P_1}{\sqrt{P_1^2 - P_2}}\right)\right), & P_2 < 0 \end{cases}$$

$$P_1 = \left(\frac{1}{27}c_1'^2 + \frac{9}{20}II_S - \frac{2}{3}II_\Omega\right)c_1',$$

$$P_2 = P_1^2 - \left(\frac{1}{9}c_1'^2 + \frac{9}{10}II_S + \frac{2}{3}II_\Omega\right)^3.$$

Summary of the model

- The 2D EARS model of Wallin & Johansson (2000) with *k*-*ε* platform.

①

$$\frac{\partial k}{\partial t}+U_j\frac{\partial k}{\partial x_j}=\underbrace{-\overline{u_i'u_j'}\frac{\partial U_i}{\partial x_j}}_{\mathcal{P}}-\varepsilon+\underbrace{\frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j}\right)}_{\mathcal{D}^{(k)}},$$

$$\frac{\partial \varepsilon}{\partial t}+U_j\frac{\partial \varepsilon}{\partial x_j}=\left(C_{\varepsilon 1}\mathcal{P}-C_{\varepsilon 2}\varepsilon\right)\frac{\varepsilon}{k}+\underbrace{\frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_j}\right)}_{\mathcal{D}^{(\varepsilon)}}.$$

②

$$\mathbf{S}=S_{ij}=\frac{1}{2}\frac{k}{\varepsilon}\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right),$$

$$\boldsymbol{\Omega}=\Omega_{ij}=\frac{1}{2}\frac{k}{\varepsilon}\left(\frac{\partial U_i}{\partial x_j}-\frac{\partial U_j}{\partial x_i}\right),$$

③

$$II_S=S_{ij}S_{ji}\qquad \mathbf{T}^{(1)}=\mathbf{S}$$

$$II_\Omega=\Omega_{ij}\Omega_{ji}\qquad \mathbf{T}^{(4)}=\mathbf{S}\boldsymbol{\Omega}-\boldsymbol{\Omega}\mathbf{S}$$

④

$$P_1=\left(\frac{1}{27}c_1'^2+\frac{9}{20}II_S-\frac{2}{3}II_\Omega\right)c_1',$$

$$P_2=P_1^2-\left(\frac{1}{9}c_1'^2+\frac{9}{10}II_S+\frac{2}{3}II_\Omega\right)^3.$$

$$N=\begin{cases}\frac{c_1'}{3}+\left(P_1+\sqrt{P_2}\right)^{1/3}+\operatorname{sign}\left(P_1-\sqrt{P_2}\right)\left|P_1-\sqrt{P_2}\right|^{1/3}, & P_2\geq 0\\ \frac{c_1'}{3}+2\left(P_1^2-P_2\right)^{1/6}+\cos\left(\frac{1}{3}\left(\frac{P_1}{\sqrt{P_1^2-P_2}}\right)\right), & P_2<0\end{cases}$$

⑤

$$\beta_1=-\frac{6}{5}\frac{N}{N^2-2II_\Omega},\qquad \beta_4=-\frac{6}{5}\frac{1}{N^2-2II_\Omega}.$$

⑥

$$a_{ij}=\beta_1T_{ij}^{(1)}+\beta_4T_{ij}^{(4)}$$

⑦

$$\overline{u_i'u_j'}=ka_{ij}+\frac{2}{3}k\delta_{ij}$$

Implementation

- Easy to implement in a code, which already has a two-equation model implemented.
- Trick: split the anisotropy tensor

Modify $C_\mu^{(\text{eff})}$

$$a_{ij} = -2 \underbrace{\frac{1}{2}\beta_1}_{C_\mu^{(\text{eff})}} S_{ij} + \underbrace{\beta_4(S_{ij}\Omega_{ij} - \Omega_{ij}S_{ij})}_{a_{ij}^{(\text{ex})}}$$

Only new term

- In the mom'm eq:

$$-\frac{\partial \overline{u_i' u_j'}}{\partial x_j} = -\frac{\partial\left(a_{ij}k + \frac{2}{3}k\delta_{ij}\right)}{\partial x_j} = \underbrace{\frac{\partial 2C_\mu^{\text{eff}} k S_{ij}}{\partial x_j}}_{\text{Treat implicit}} - \boxed{\underbrace{\frac{\partial a_{ij}^{(\text{ex})} k}{\partial x_j}}_{\text{Treat explicit}}} - \underbrace{\frac{\partial \frac{2}{3}k\delta_{ij}}{\partial x_j}}_{\text{Absorb into pressure}} .$$

- Remember to change shear production in transport equations

$$\xcancel{\mathcal{P} = \nu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j}} \qquad \mathcal{P} = -\varepsilon a_{ik} S_{ki} . \; ✓$$

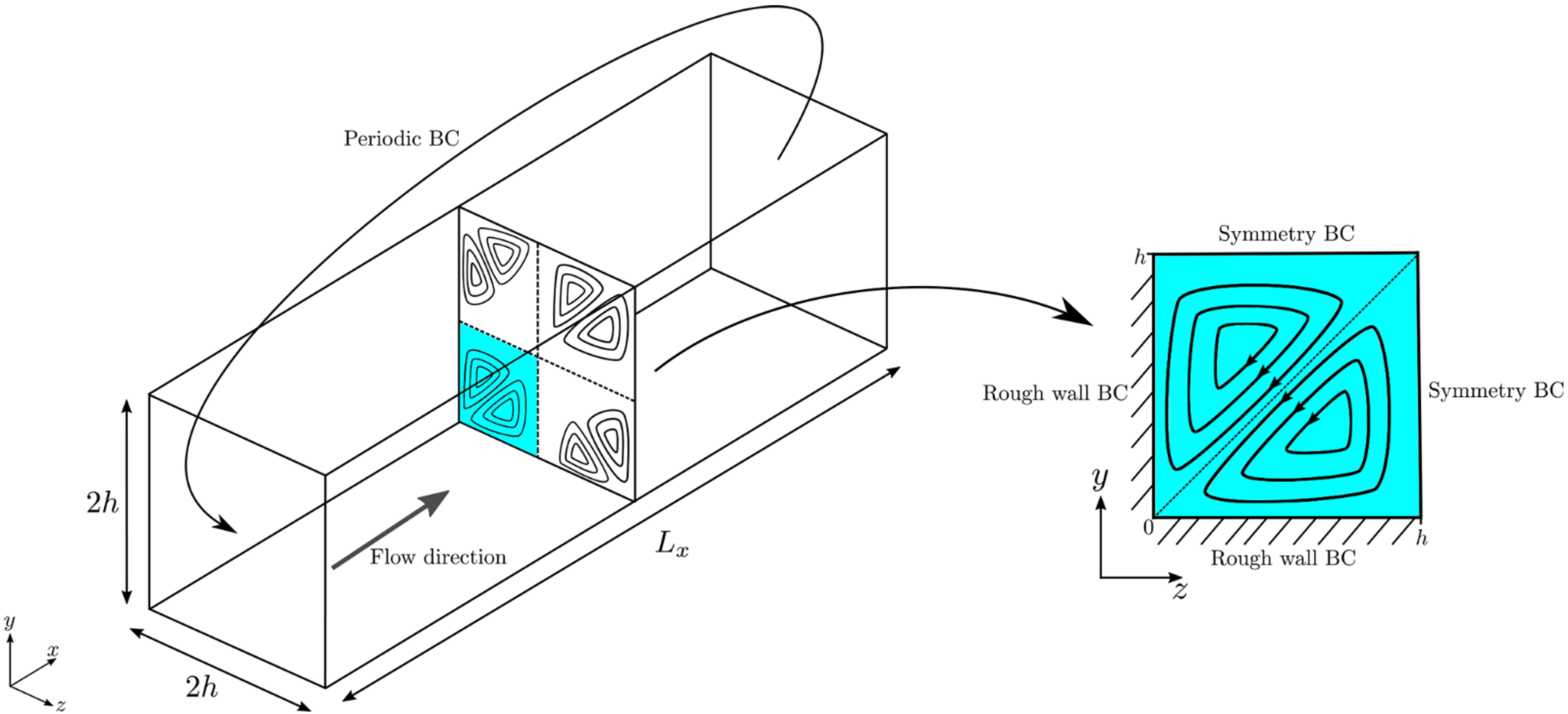
Application of EARS models

Codes with the WJ-EARS model

- Maple/FORTRAN 1D code (KTH)
- Python 1D code (KTH)
- Edge (FOI/KTH)
- FINFLO (Helsinki University)
- TAU (DLR)
- EllipSys1D and EllipSys3D (DTU)
- Ansys CFX (manual)
- OpenFOAM user model (implemented for OF1.7.x)
- …

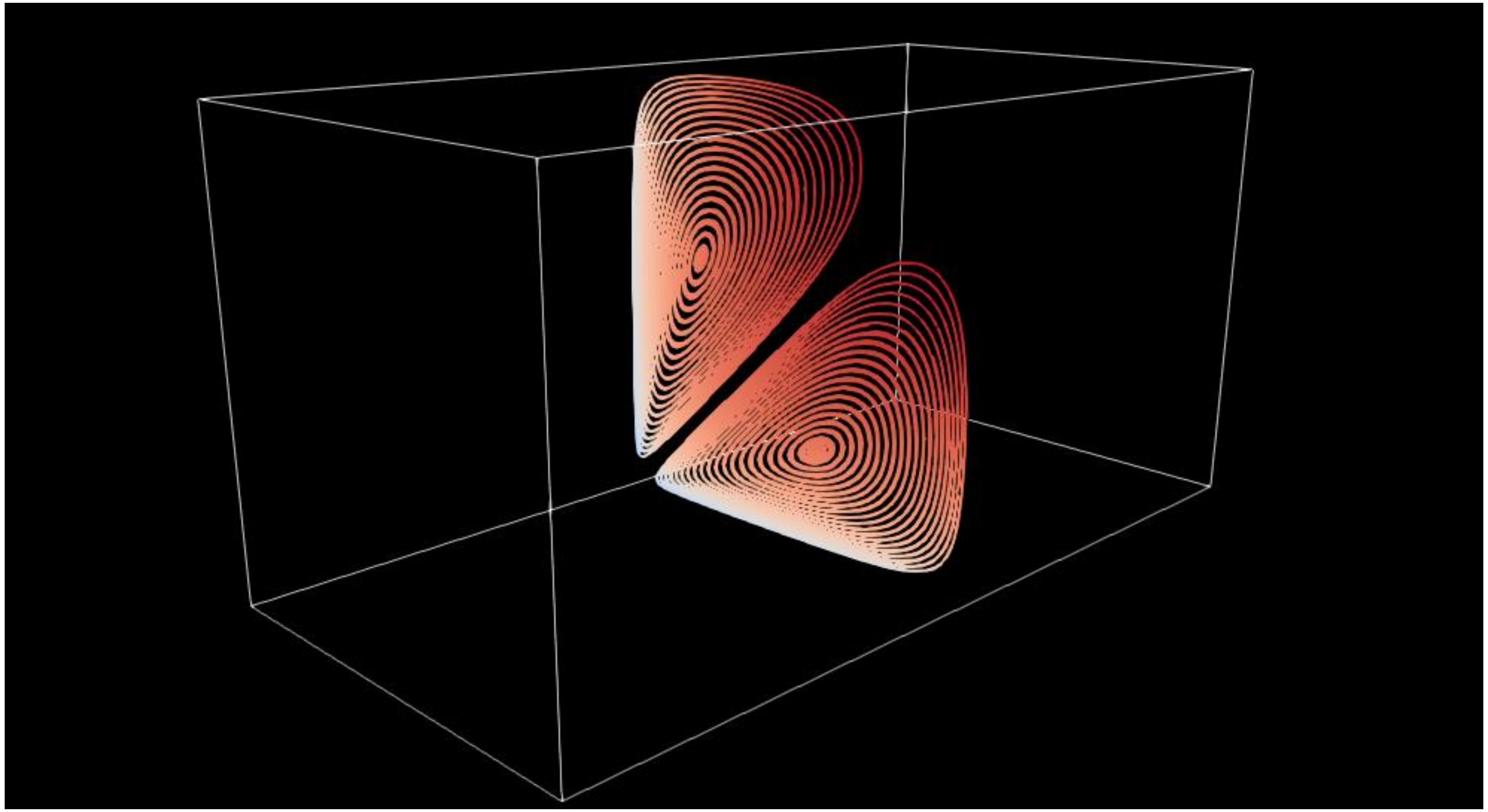
Square duct flow (1/3)

- Secondary flow in corners

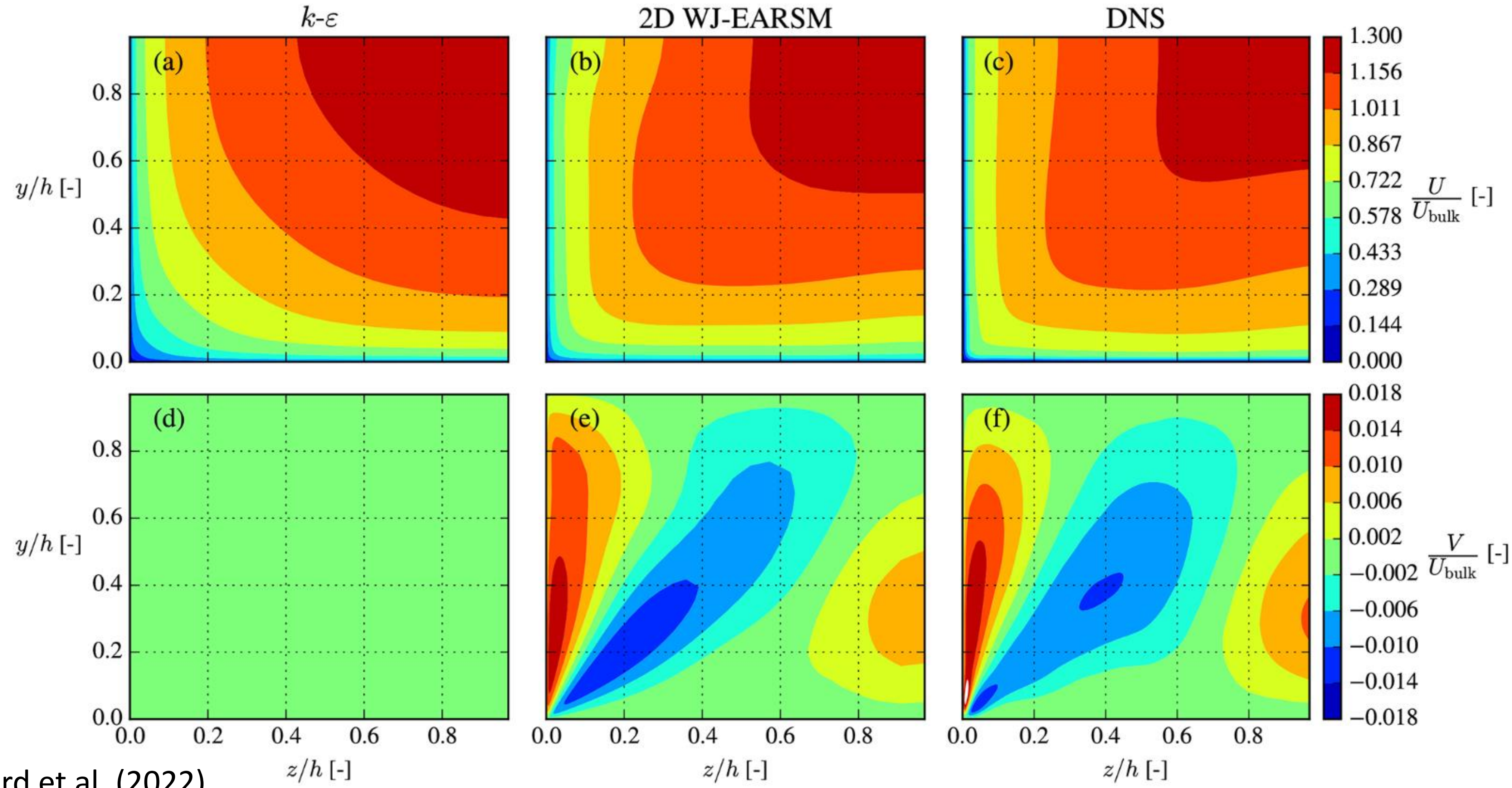


Square duct flow (2/3)

- Simulation of lower left quadrant in EllipSys3D with WJ-EARS model



Square duct flow (3/3)

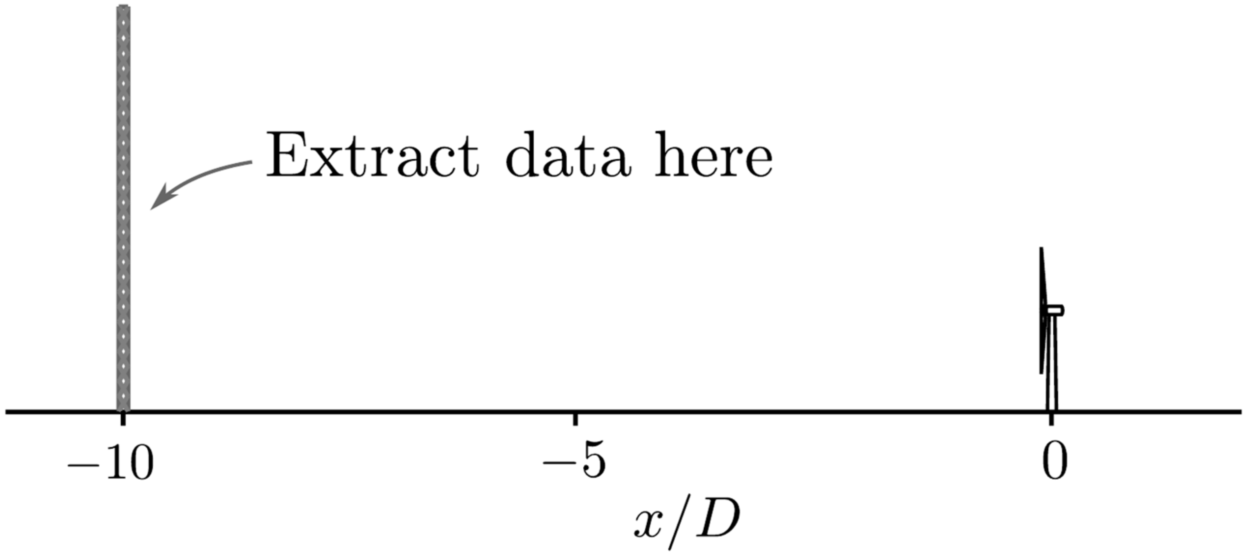


Baungaard et al. (2022)

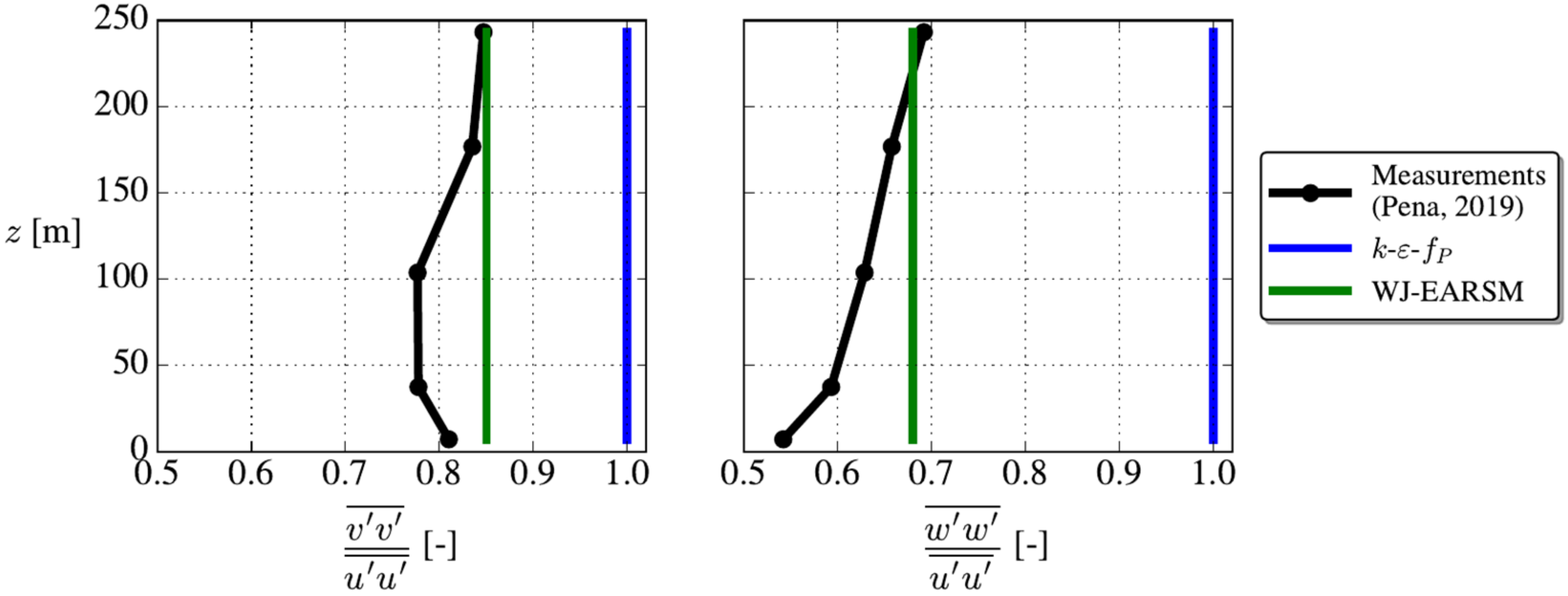
A neutral ABL (1/2)

$$k \equiv \frac{1}{2}(\overline{u'u'} + \overline{v'v'} + \overline{w'w'})$$

What is the distribution of TKE among its components in the neutral atmospheric boundary layer (ABL)?



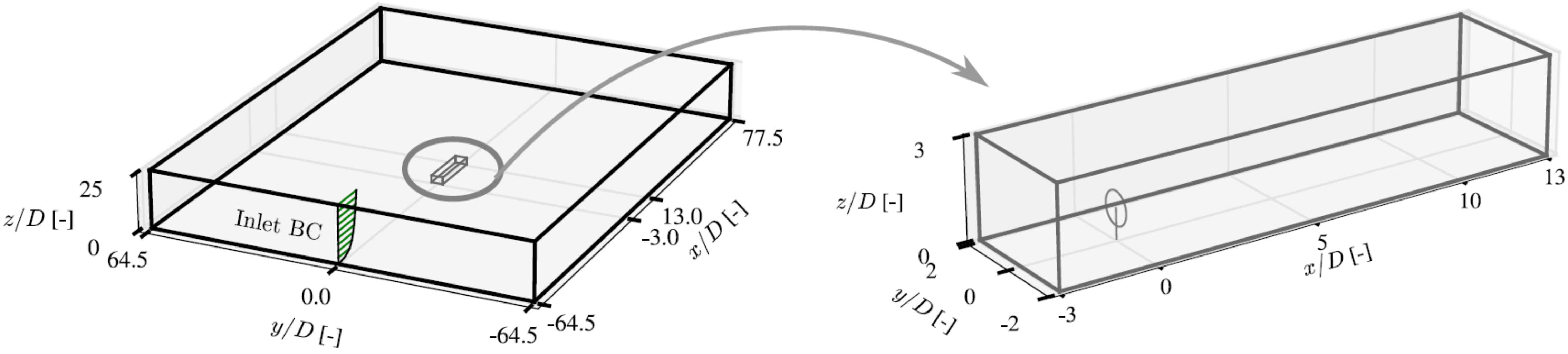
A neutral ABL (2/2)



In the ABL, $\overline{u'u'} > \overline{v'v'} > \overline{w'w'}$

Impossible to capture with linear EVMs!

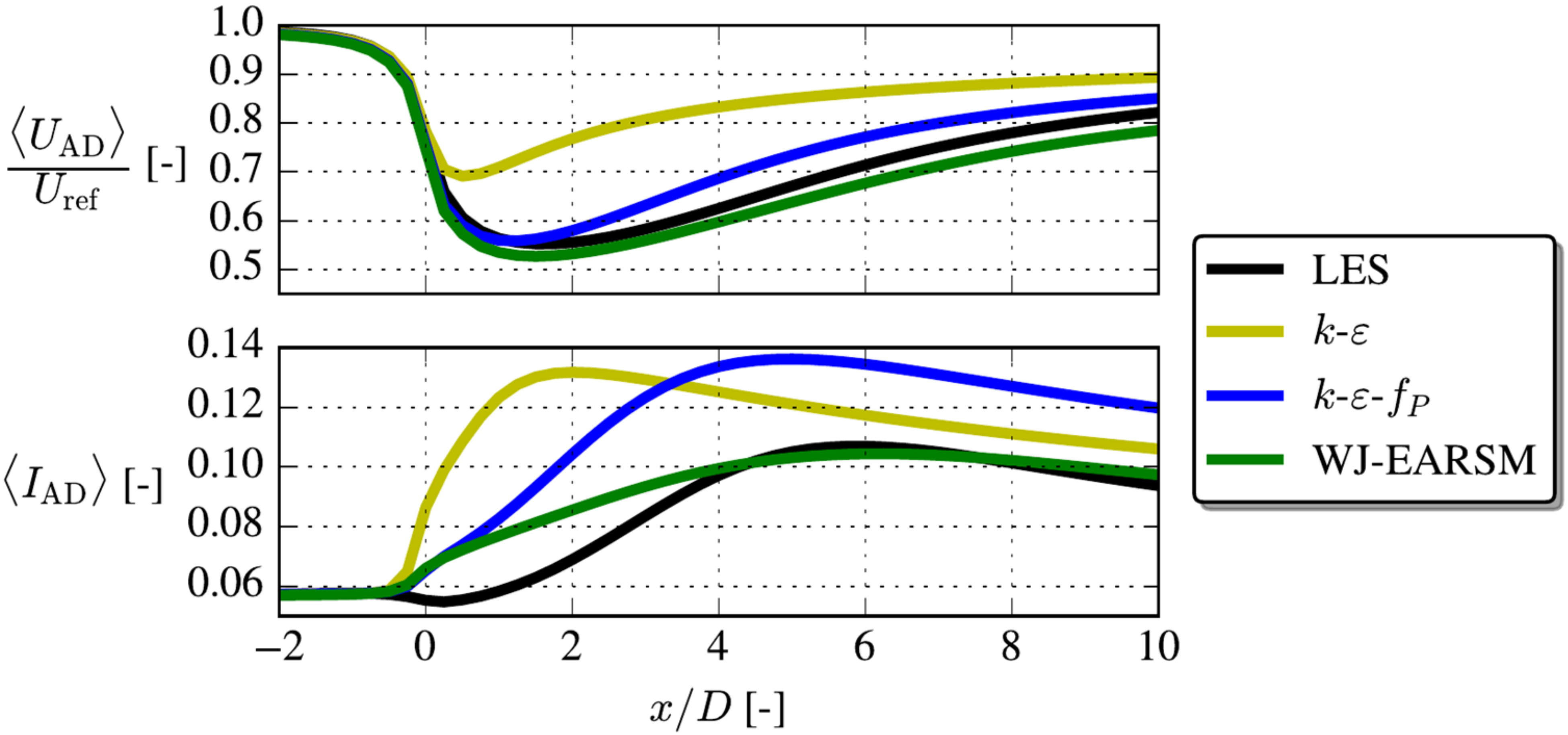
A neutral ABL + a wind turbine (1/4)



- EllipSys3D code
- V80 turbine modelled as AD
- Grid stretching to avoid excessive amount of cells

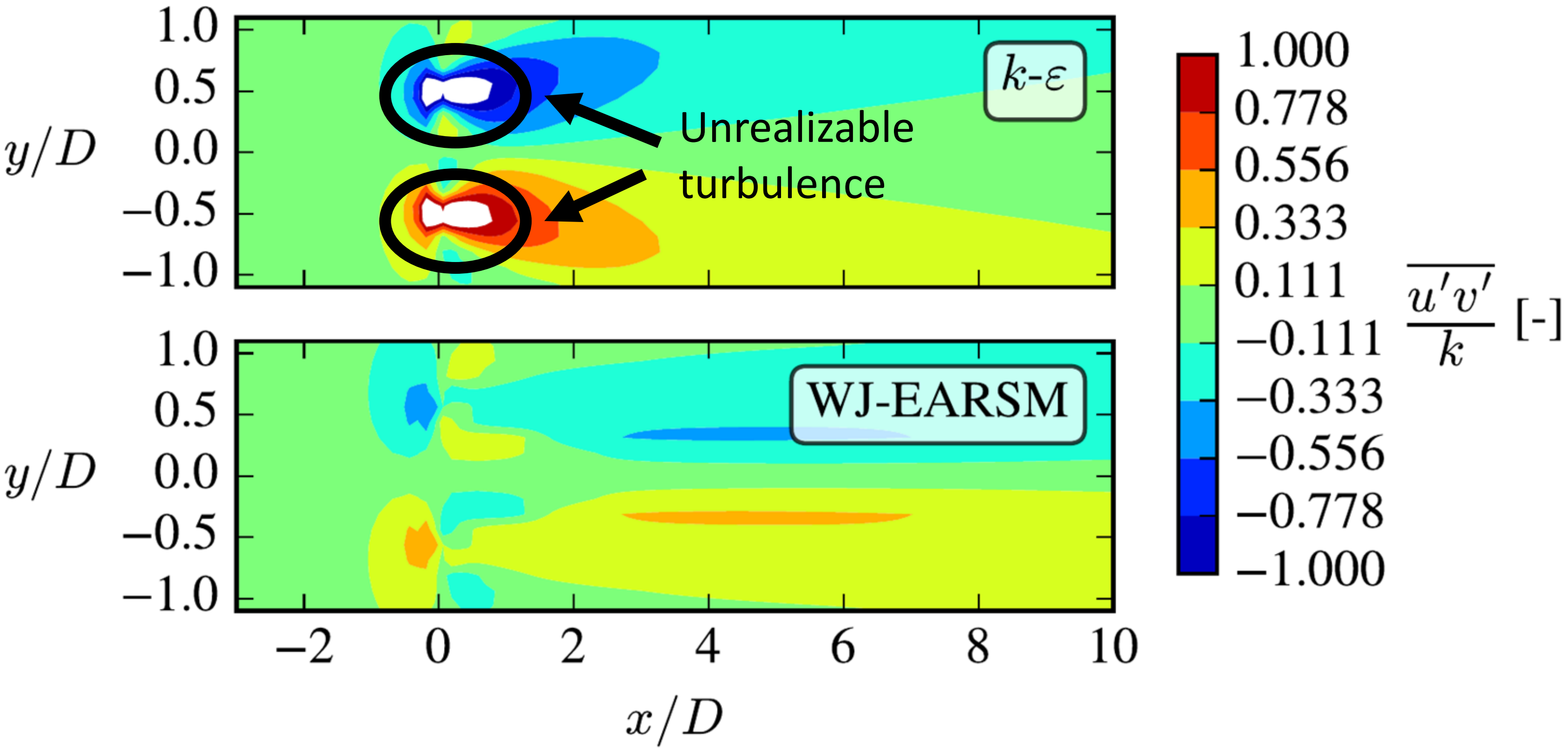
A neutral ABL + a wind turbine (2/4)

- Comparison of disk-averaged quantities with LES data



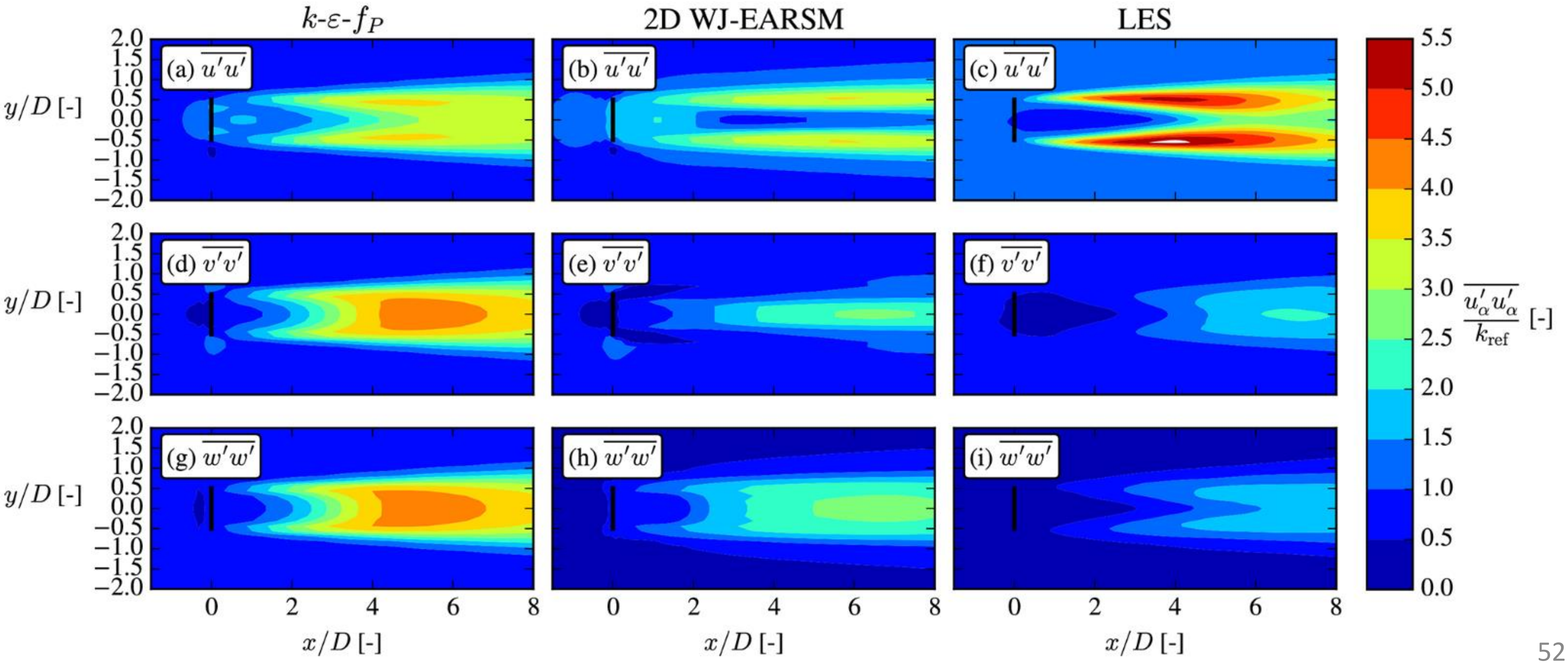
A neutral ABL + a wind turbine (3/4)

- Why does *k*-*ε* overpredict wake recovery?
  Because shear stress is overpredicted.



A neutral ABL + a wind turbine (4/4)

- Linear EVM (k-ε-f_P) has too isotropic wake turbulence



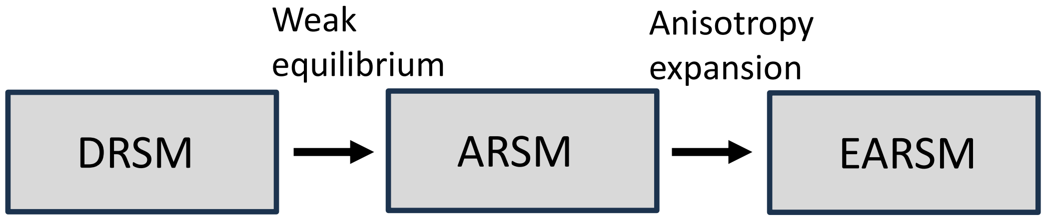
Summary

Linear eddy-viscosity models (EVMs)

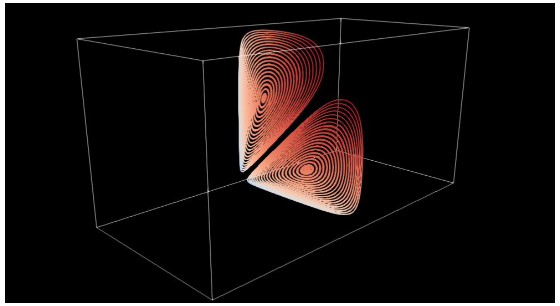
$$a_{ij} = -2C_\mu^{(\text{eff})} S_{ij}$$

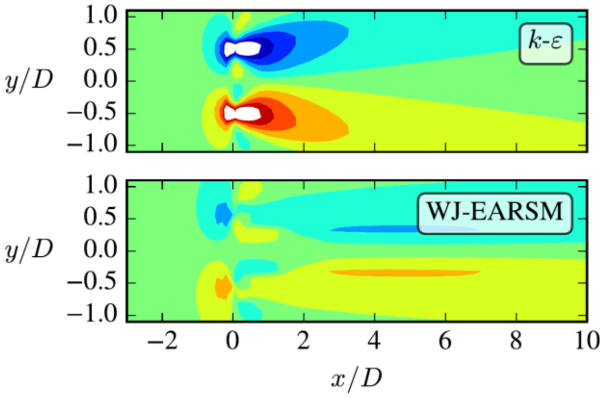
	Linear EVM	WJ-EARSM
Anisotropic freestream turbulence	❌	✅
Secondary flows	❌	✅
Counter-gradient heat fluxes	❌	✅
Realizable turbulence	Some	✅
Sensitive to rotation	Very few	✅

Theory of EARS models



Applications of EARS models





Takeaways

<u>What is an EARS model?</u>

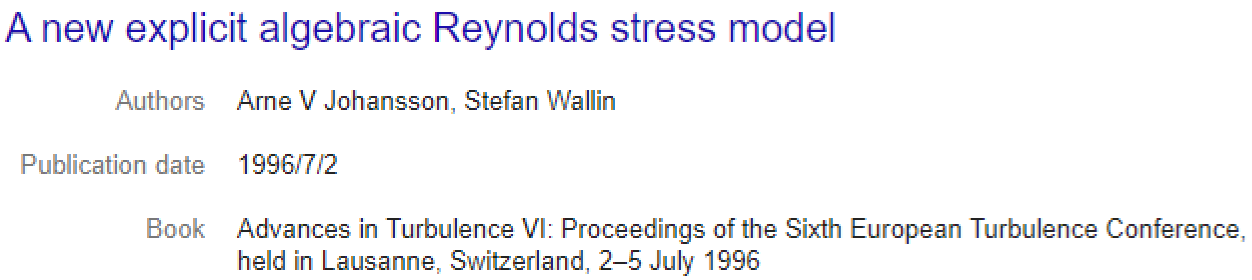
- A turbulence model derived directly from the Reynolds stress equations.
- Does not rely on the Boussinesq hypothesis!

<u>Why use an EARS model?</u>

✓ Relatively easy to implement.
✓ Only slightly more computationally expensive than linear EVMs.
✓ Automatically includes more physics than linear EVMs.
✓ More numerically robust than DRSMs.

The end

A good introduction (4 pages):



A new explicit algebraic Reynolds stress model

Authors	Arne V Johansson, Stefan Wallin
Publication date	1996/7/2
Book	Advances in Turbulence VI: Proceedings of the Sixth European Turbulence Conference, held in Lausanne, Switzerland, 2–5 July 1996

https://link.springer.com/chapter/10.1007/978-94-009-0297-8_8

Some things are easier to discuss at a blackboard!
We can talk more about EARS models:

Wednesday 7 Aug @ 15.00
@ Lecture room 7

https://mchba.github.io/240731_intro_to_earsm.pdf

Slides are here! →

Extra slides

Realisability: limits of the anisotropy tensor

Rule 1 Rule 2

Greek indices imply that there is <u>no</u> summation over repeated indices!

$$-\frac{2}{3} \leq a_{\alpha\alpha} \leq \frac{4}{3},$$

$$-1 \leq a_{\alpha\beta} \leq 1.$$

Rule 3 Rule 4

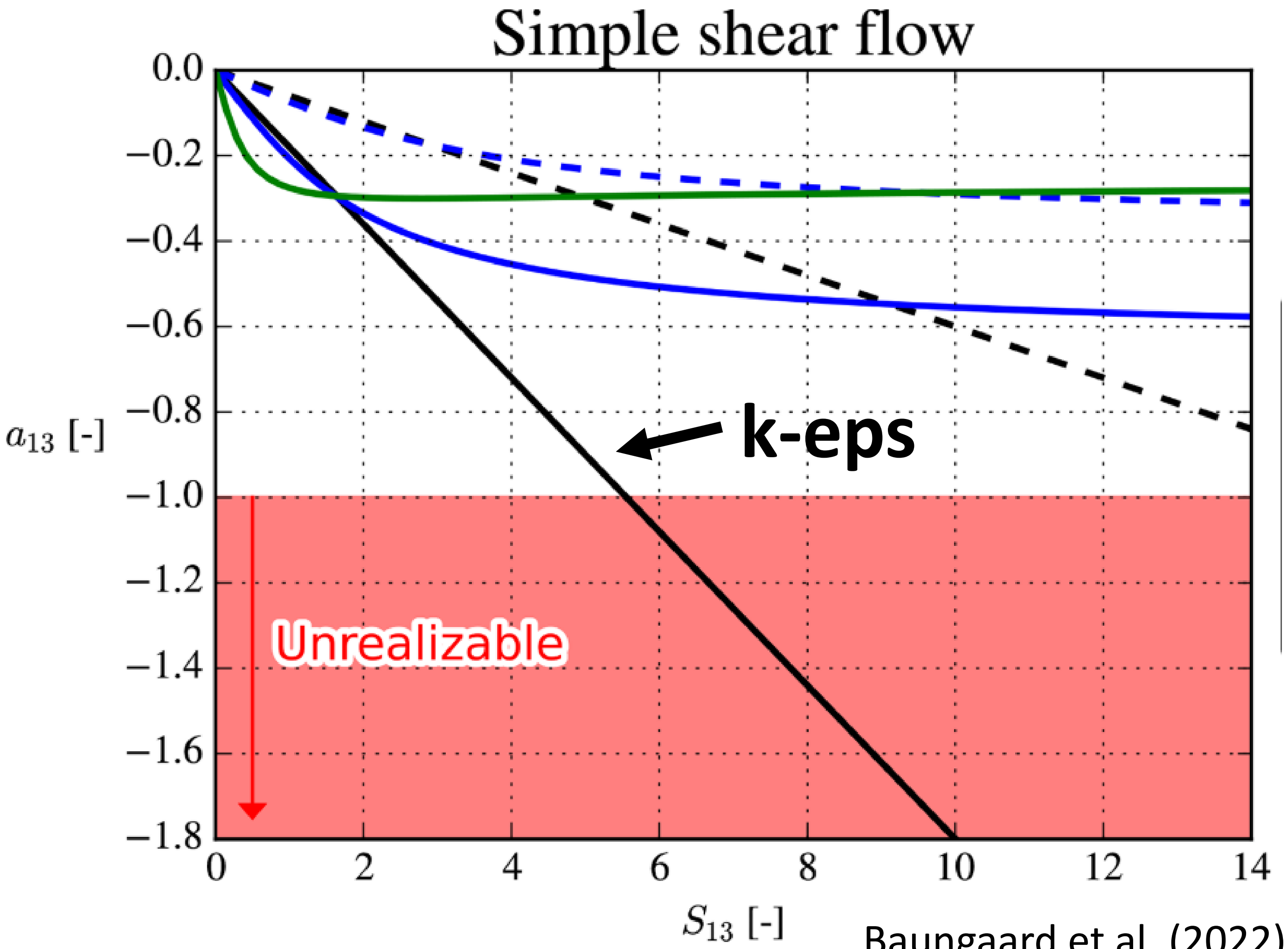
- Rule 1: Each normal stress must be positive.
- Rule 2: From the definition of TKE.
- Rule 3: Each shear stress must satisfy the Cauchy-Schwarz inequality.
- Rule 4: Comes from rule 2.

k-eps can give unrealizable turbulence

Example: simple shear flow

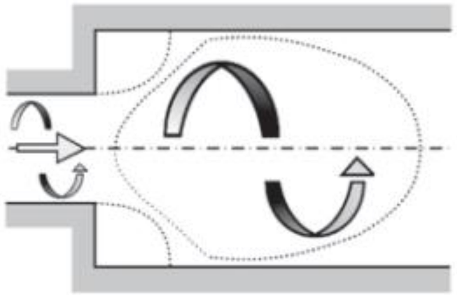
$$\mathbf{S}_{\text{simple}} = \begin{pmatrix} 0 & 0 & S_{13} \\ 0 & 0 & 0 \\ S_{13} & 0 & 0 \end{pmatrix},$$

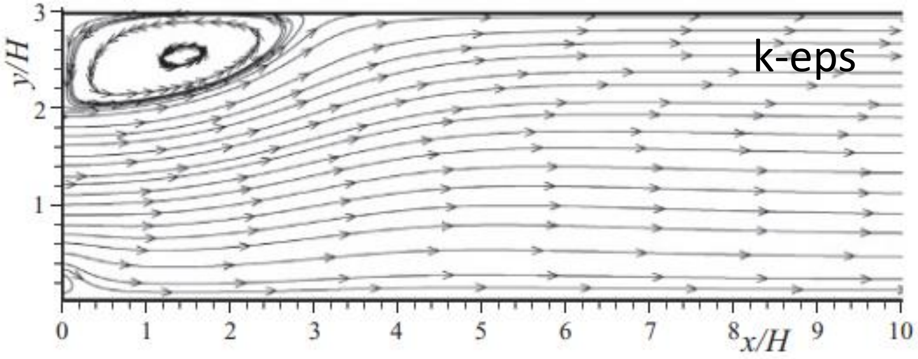
$$\mathbf{a}_{k\text{-}\varepsilon} = -2C_\mu \mathbf{S}_{\text{simple}}$$



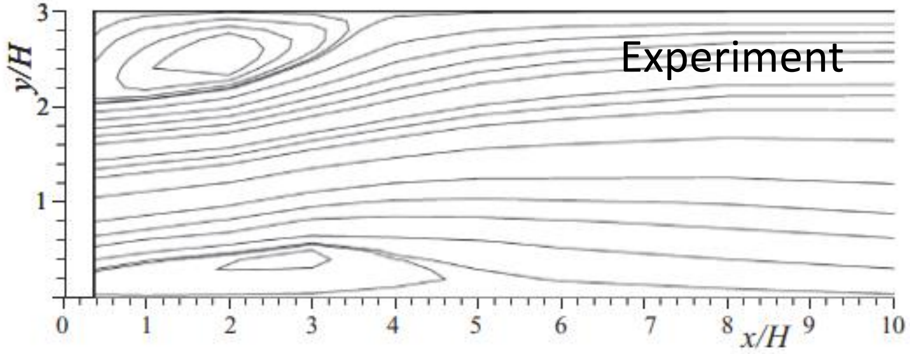
Baungaard et al. (2022)

Example 2: swirling flow in an expanding pipe

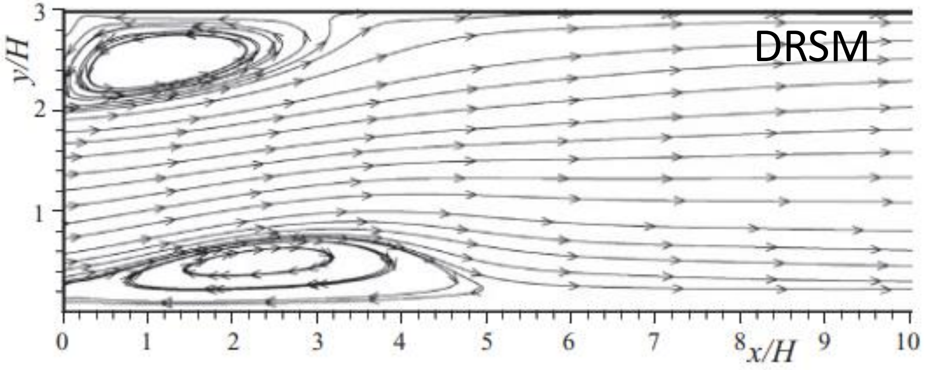




k-eps



Experiment



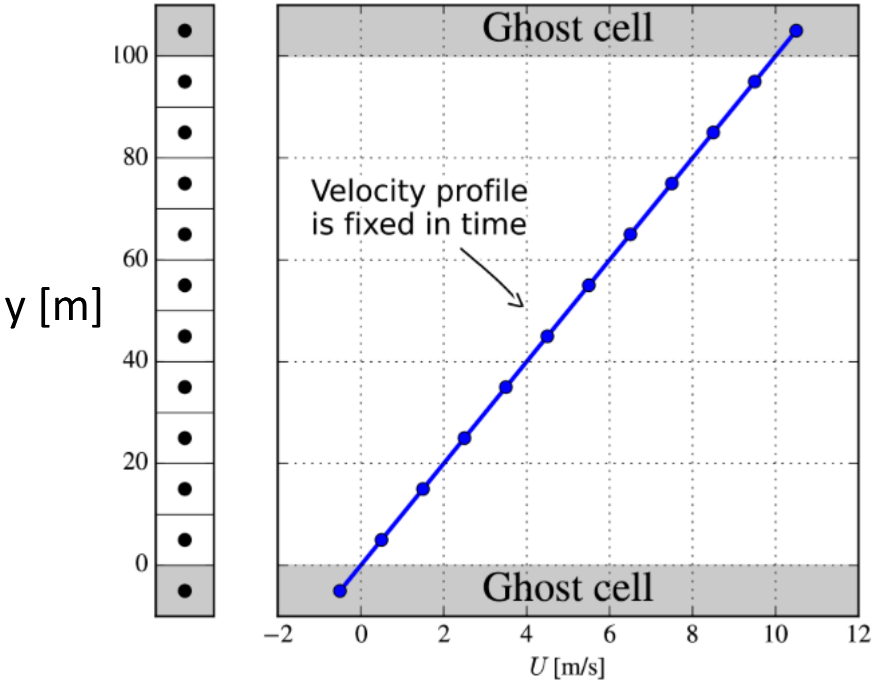
DRSM

Hanjalic & Launder (2011, p.81)

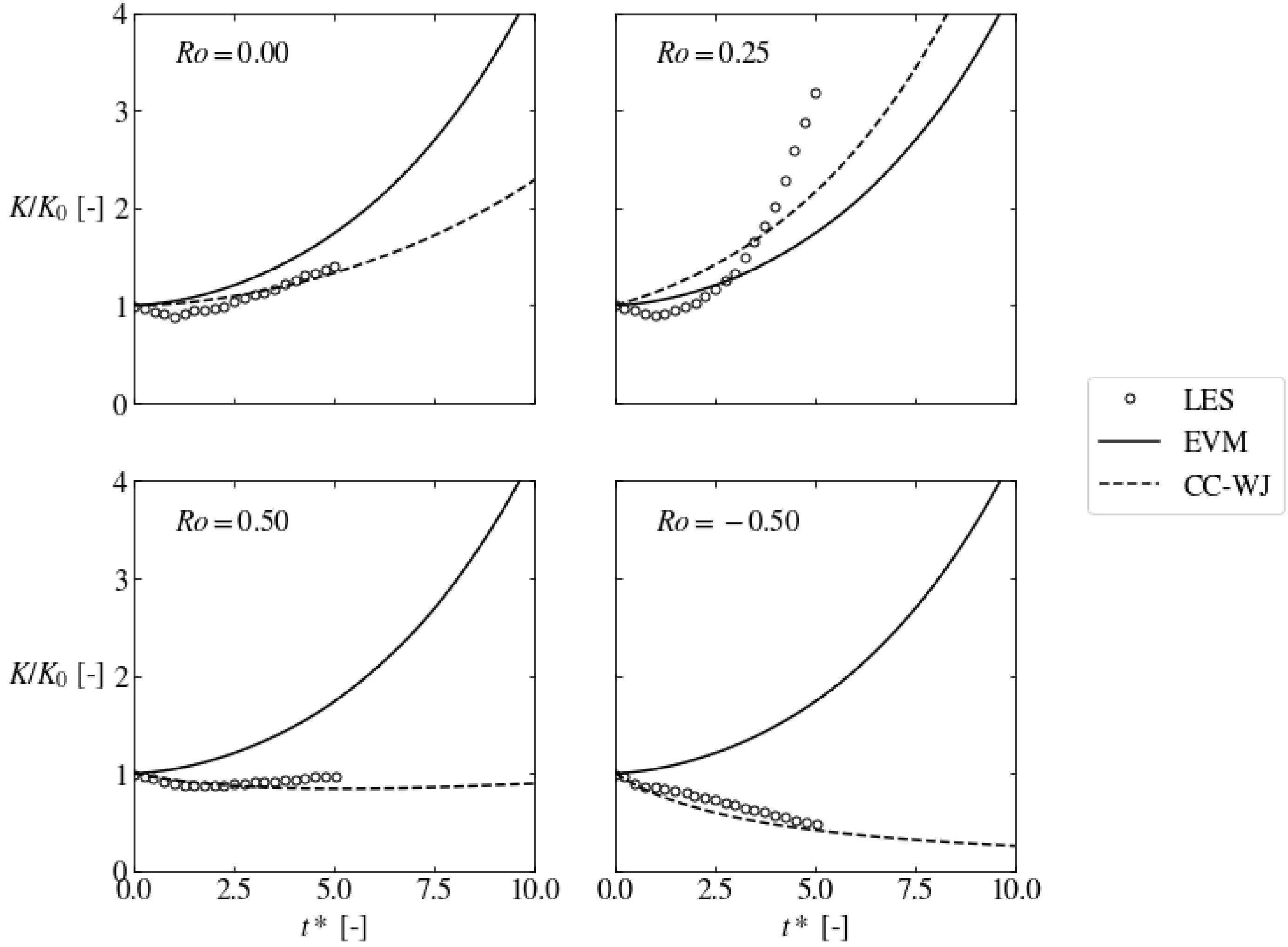
Rotating homogeneous shear flow

- A simple first testcase

Setup in rotating frame of reference:



Velocity profile is fixed in time

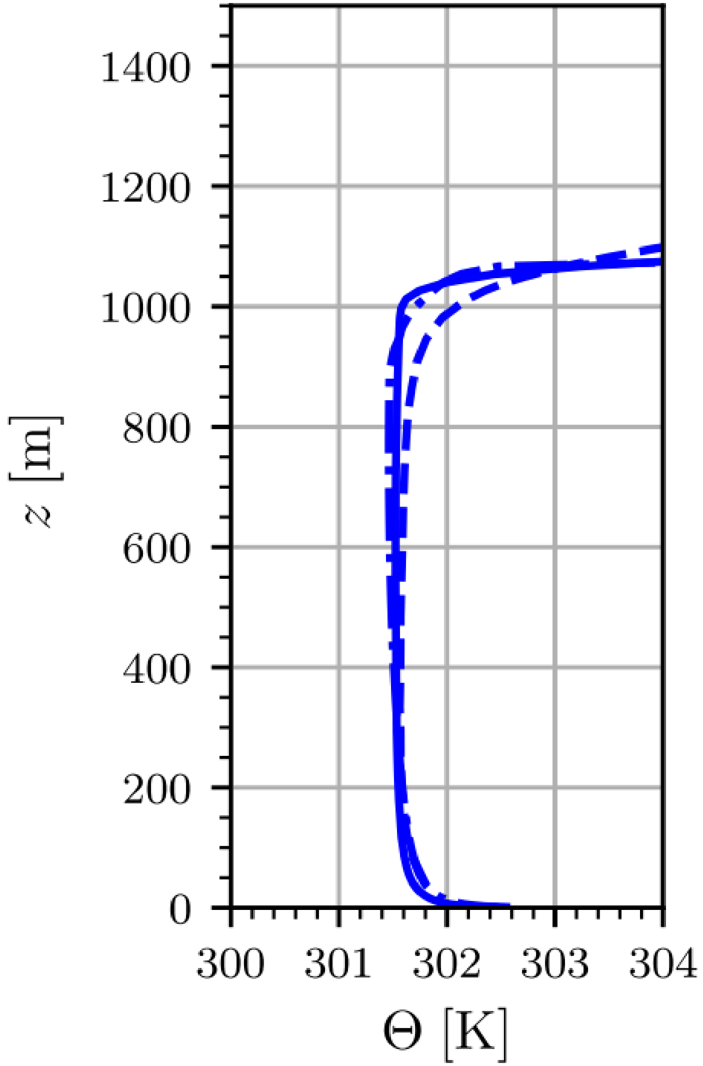


Rotation number:

$$Ro \equiv \omega_z^{(\mathrm{r})}/(\mathrm{d}U/\mathrm{d}y)$$

A convective ABL

Zeli et al. (2021)

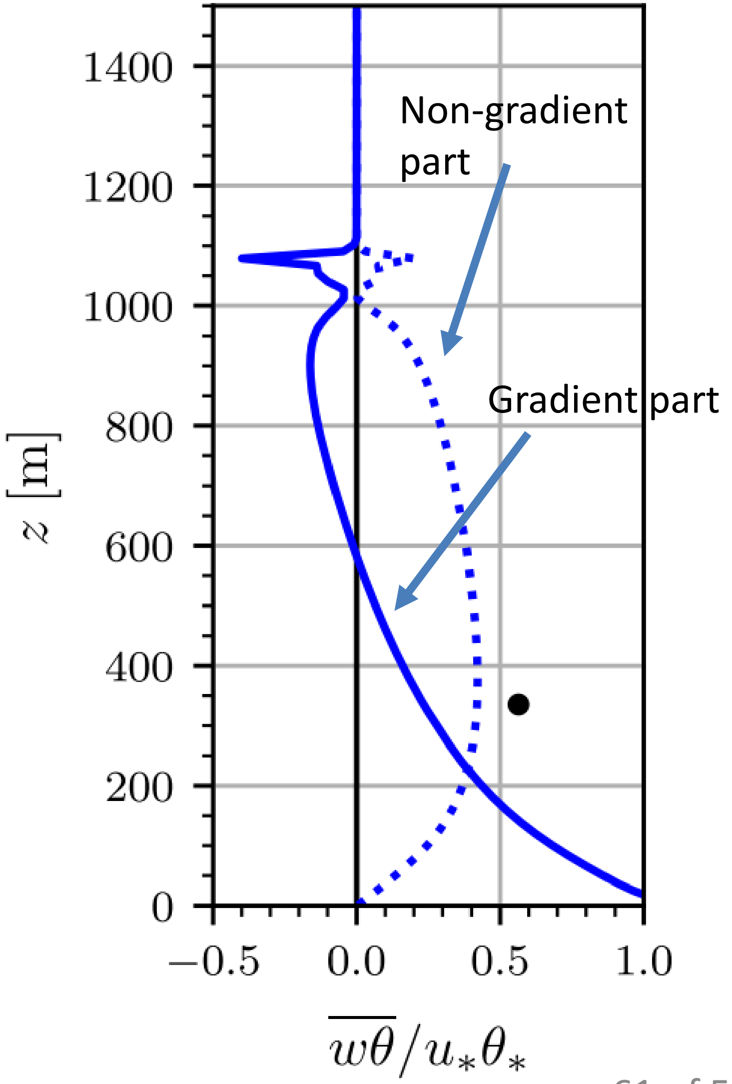


Linear EVM:

$$\overline{u_i'\theta} = -\frac{\nu_t}{Pr_t}\frac{\partial\Theta}{\partial x_i}$$

EARSM:

$$\overline{u_i'\theta} = -\frac{\nu_t^{(\mathrm{eff})}}{Pr_t^{(\mathrm{eff})}}\frac{\partial\Theta}{\partial x_i} + \Phi_i$$



Non-gradient part

Gradient part